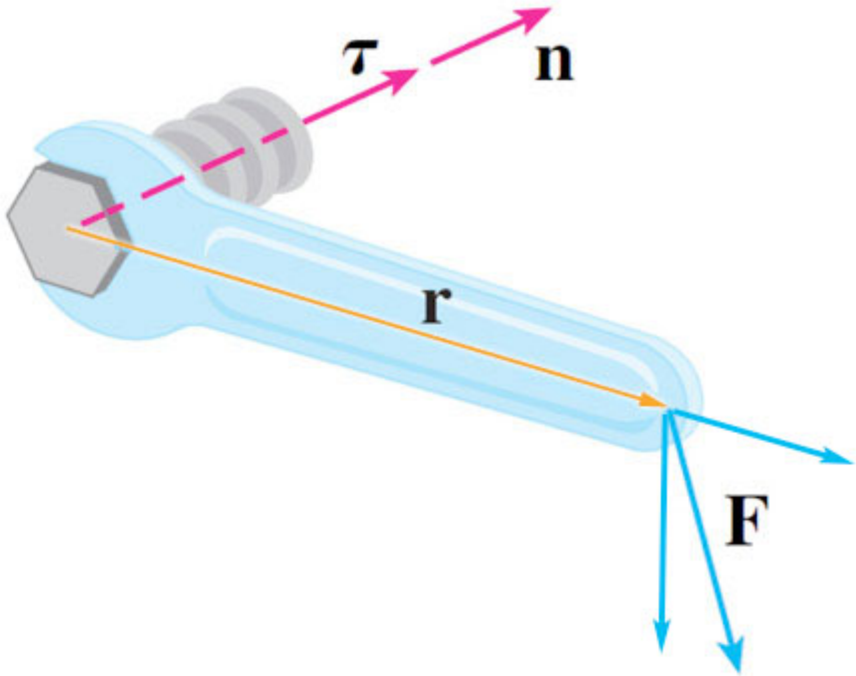
Cross product [9.4]

Torque

Or, how much of a force contributes to tightening a bolt?

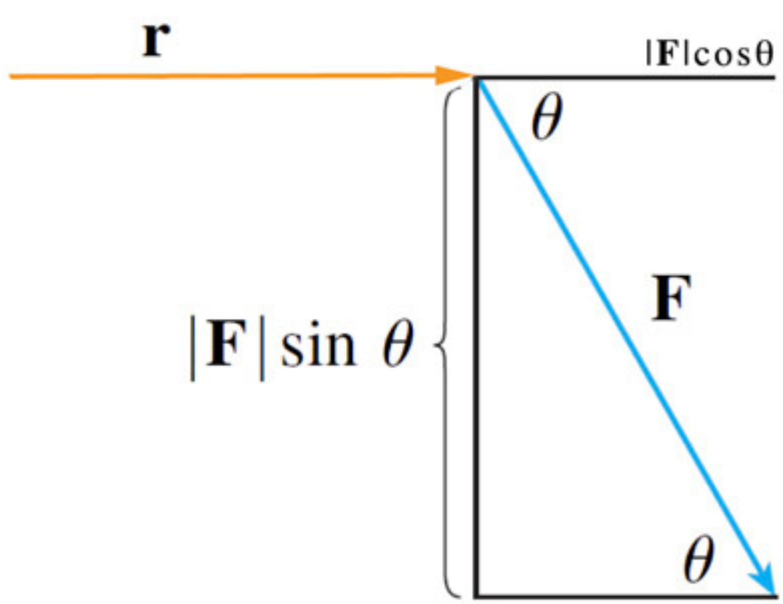
The "**torque**", τ, necessary to tighten a bolt is related to:

- The force, F, exerted on the wrench,
- the distance, r, from the axis of the bolt to the point where the force is applied, and
- the relative orientation of $\vec{\mathbf{F}}$ and $\vec{\mathbf{r}}$.



Meaning of the direction of $\vec{\tau}$? $\vec{\tau}$ is parallel to the axis of the bolt. $\vec{\tau}$ points in the direction that a standard, right-hand threaded bolt would advance when tightened.

Which component of the force--parallel or perpendicular to $\vec{\mathbf{r}}$--is more effective at tightening bolts?



We conclude that

$$\tau = rF\sin\theta$$

to motivate the...

Definition of the cross product

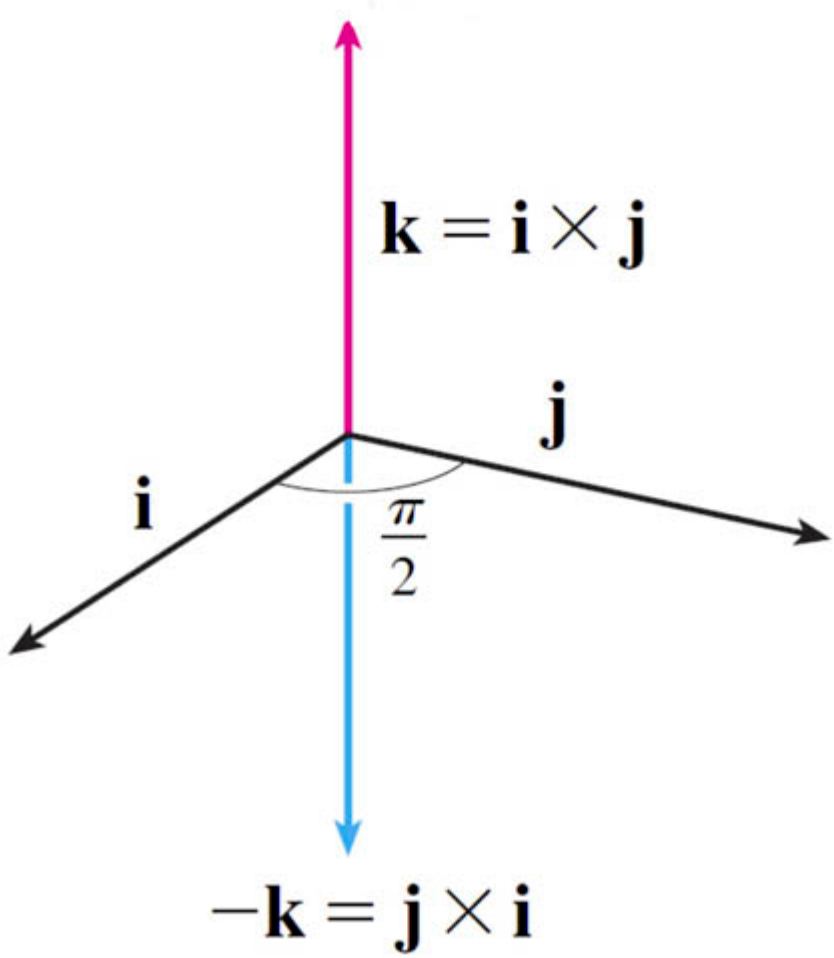
If $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are non-zero 3-d vectors, then the cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \equiv |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta \, \hat{\mathbf{n}}, \tag{1}$$

where θ is the smaller angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0 \leq \theta \leq \pi)$ and $\hat{\mathbf{n}}$ is a unit vector perpendicular to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ according to the right-hand rule: Curl r.h. fingers from $\vec{\mathbf{a}}$ towards $\vec{\mathbf{b}}$ and your thumb points in the same direction as $\hat{\mathbf{n}}$.

Cross product: order

With the right-hand rule, the **order** of the vectors in the cross product **matters**.



Properties

Cross Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

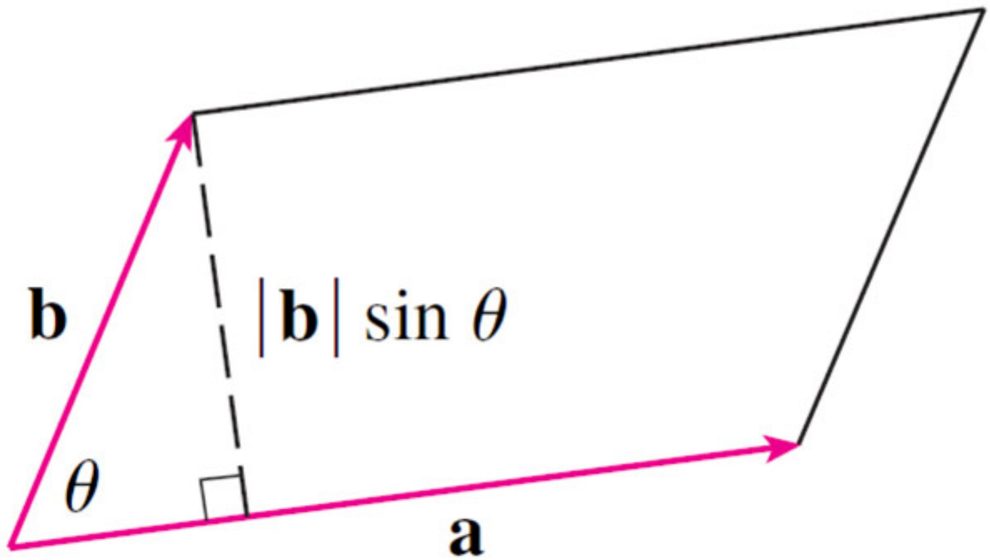
$\mathbf{b} \times \mathbf{a}$

$c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

$= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

$= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

Geometry



The length of the cross product, $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$, is the area of the parallelogram determined by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

See www.geogebra.org/m/hjgxmz5a

Computing the cross product

Grinding out the cross product from its components...

$$\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

$$b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k}$$

$$+ a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k}$$

$$+ a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k}$$

$$b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} + a_3b_2(-\mathbf{i})$$

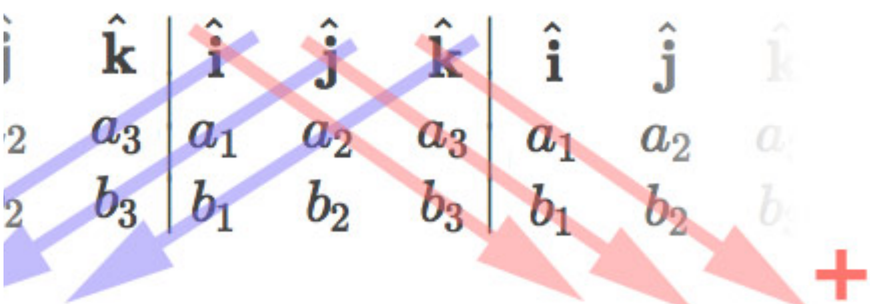
$$b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Cross product: determinant method

I remember how to compute the cross product by using the determinant of this 3×3 matrix:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \det \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \qquad (2)$$

To compute the determinant, add the red products, and subtract the blue products



$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_2b_3 - a_3b_2)\,\hat{\mathbf{i}} + (a_3b_1 - a_1b_3)\,\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\,\hat{\mathbf{k}}.$$

For example: $\vec{\mathbf{a}} = \langle 1, -2, -4 \rangle$ and $\vec{\mathbf{b}} = \langle 2, 4, 8 \rangle$

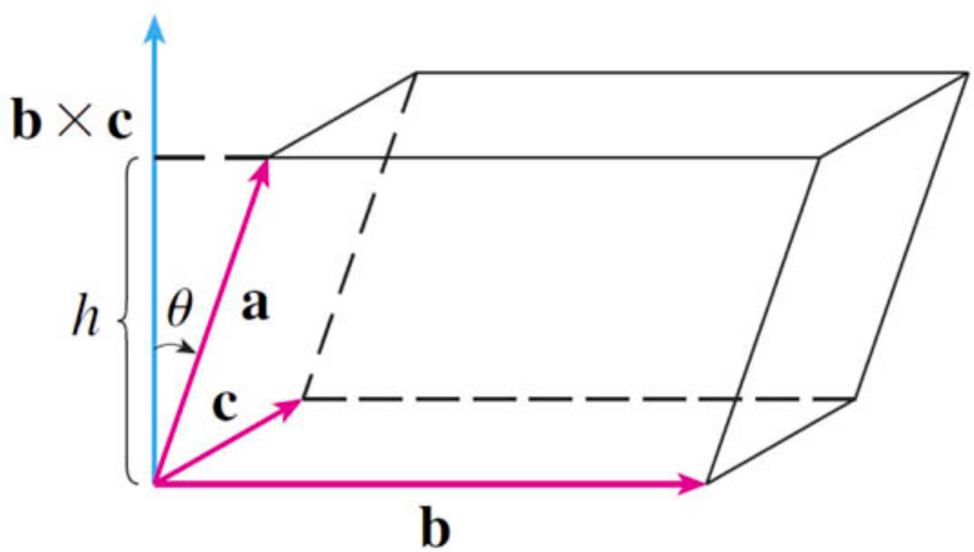
$$
\begin{aligned}
\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \det & \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2 & -4 \\ 2 & 4 & 8 \end{vmatrix} \\
= & \\
= & \\
= & (-2*8-(-4*4))\,\hat{\mathbf{i}} + (-4*2-(1*8))\,\hat{\mathbf{j}} + (1*4-(-2*2))\,\hat{\mathbf{k}} \\
= & -16\,\hat{\mathbf{j}} + 8\,\hat{\mathbf{k}} = \langle 0, -16, 8 \rangle.
\end{aligned} \tag{3}
$$

Triple products

There are many ways to form triple products. Of these, which are vector or scalar quantities?

1. $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})$
2. $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$
3. $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})$
4. $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$

Scalar triple product



The volume of the parallelipipid determined by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ is:

$$V = hB \tag{4}$$

Where B is the area of the base (defined by $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$) and h is the height of the parallelipid above its base. [Are extremes plausible...?] Now...

- height is $h = |\vec{\mathbf{a}}| \cos\theta$ where θ is the angle to the vector direction which is perpendicular to the base.

- area of the base is $B = |\vec{\mathbf{b}} \times \vec{\mathbf{c}}|$.

  ...and $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is a vector perpendicular to the base, as shown, with the magnitude of B.

  So, the volume can be written as

$$V = |\vec{\mathbf{a}}| \cos\theta |\vec{\mathbf{b}} \times \vec{\mathbf{c}}| = \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}). \tag{5}$$

  [Without proof] This may also be calculated from:

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{6}$$

Vector triple product

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}. \tag{7}$$

Perpendicular to $\vec{\mathbf{a}}$

Consider the set of all vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ such that $\vec{\mathbf{b}} \times \vec{\mathbf{c}} = k\vec{\mathbf{a}}$. (a scalar multiple of $\vec{\mathbf{a}}$). What geometrical entity do these vectors form?

To do

- Messing around with the cross product.