Lines and planes [9.5]

Casa Batlló by Antoni "enemy of the straight line" Gaudí and Josep Maria Jujol.

Line segment

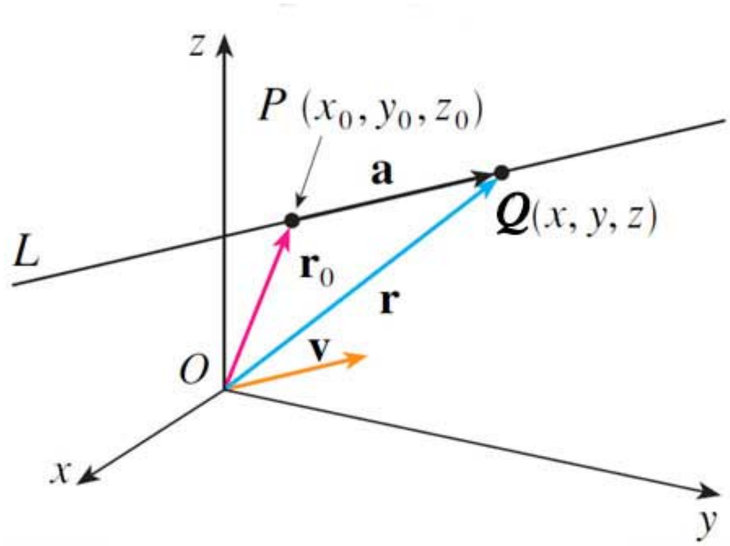
You know that "two points determine a line".

In 3-d we should be able to construct a line given:
2 points $\times$ 3 coordinates / pt = 6 numbers.

In yesterday's class, you *may* have figured out (in two dimensions) that this equation specifies *all* the points on the **line segment** between P and Q, where the vector from P to Q is $\overrightarrow{\mathbf{PQ}} \equiv \vec{\mathbf{a}}$...

$$(\, x(t), y(t)\,) = (P_x, P_y) + t(a_x, a_y)\ :\ 0 < t < 1. \tag{1}$$

You *may* have figured out that if we lifted the restriction $0 < t < 1$, and allowed t to be *any* real number, positive or negative, that we'd have the **equation of the line** that runs through both P and Q!



Translating this into the terms of the diagram above (and generalizing to 3 dimensions...)

- $\vec{\mathbf{r}}_0$ is the **position vector** with tail at the origin, and head at point P.
- $\overrightarrow{\mathbf{PQ}} \equiv \vec{\mathbf{a}}$ is the difference vector, which runs from P to some other point Q
- Let's say that $\vec{\mathbf{v}}$ is some other vector, that is parallel to $\vec{\mathbf{a}}$. *Any* vector parallel to $\vec{\mathbf{v}}$ can be expressed as some scalar multiple $t\vec{\mathbf{v}}$ of v.

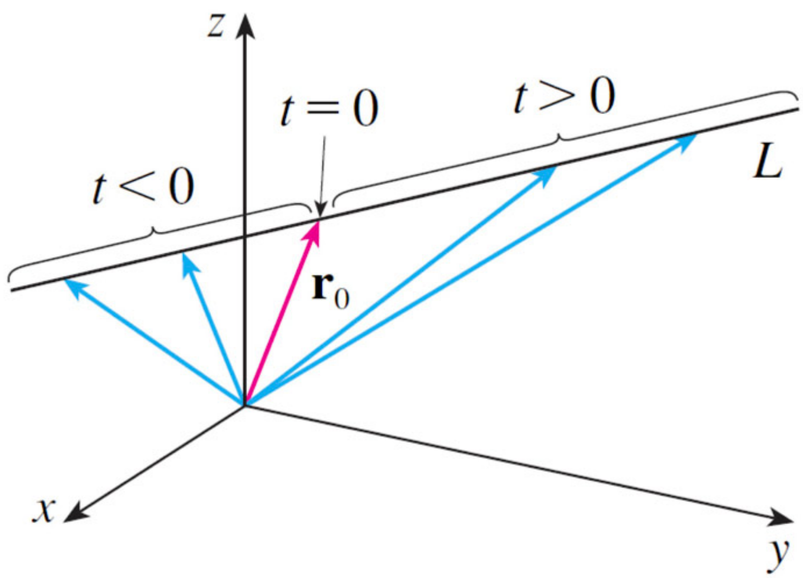
So, using vector addition, we can express the set of points on the line in parametric form as:

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}. \tag{2}$$

> **Tip: WebAssign problem 9.5.002** In part 1 of this question, they ask you for the "vector equation of a line". They're expecting
>
> $$r_0 + tv \tag{3}$$
>
> with no special vector formatting. To get the subscript, type '`r_0`'. When you type the underscore character, '_', you won't see it appear, but when you type the 0, it should show up as a subscript.

...And since $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}}$ each have three components, once again, we have the 6 scalars needed to uniquely specify a line, only packaged up this time as the coordinates of a point and the components of a vector.

In component form



$$\begin{aligned}\vec{\mathbf{r}}(t) =&\vec{\mathbf{r}}_0 + t\vec{\mathbf{v}} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle \\ =&\langle x_0 + ta, y_0 + tb, z_0 + tc\rangle.\end{aligned} \tag{4}$$

or

$$x(t) = x_0 + at; \;\; y(t) = y_0 + bt; \;\; z(t) = z_0 + ct \tag{5}$$

Each of these can be solved for t, and then setting each form of t to the others:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{6}$$

These are the **symmetric equations**. Notice that a, b, c are the components of $\vec{\mathbf{v}}$, a vector parallel to the line L.

Example

Consider two lines:

$$x = 1 + t; \;\; y = -2 + 3t; \;\; z = 4 - t$$

and

$$x = 2s; \;\; y = 3 + s; \;\; z = -3 + 4s$$

Do they intersect?

Solution: If they intersect at some common point (x,y,z), then that point should be a simultaneous solution in s and t to these three equations:

$$1 + t = 2s; \quad -2 + 3t = 3 + s; \;\; 4 - t = -3 + 4s \tag{7}$$

Those same two lines:

$$x = 1 + t; \;\; y = -2 + 3t; \;\; z = 4 - t$$

and

$$x = 2s; \;\; y = 3 + s; \;\; z = -3 + 4s$$

Are the lines parallel?

Solution: Re-arrange the equations into the form of the symmetric equations (solve each equation for t...or for s...) to find the components (denominators) of a vector parallel to each line.

Lines which are not parallel to each other, and do not intersect are called **skew lines**.

To do

- *Lines in the plane*

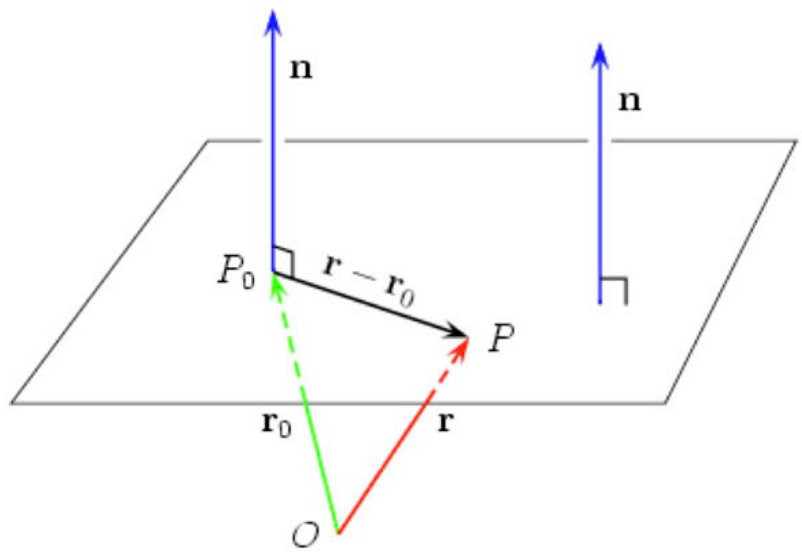
Planes

A plane can be determined by

- a point P_0 in the plane and
- a **normal vector**, $\vec{\mathbf{n}}$, which is **orthogonal** (normal) to the plane.

In pictures...

- $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0 \rangle$ is a vector pointing at P_0.
- $\vec{\mathbf{n}} = \langle a, b, c \rangle$ is the normal vector.
- $\vec{\mathbf{r}} = \langle x, y, z \rangle$: Some point in the plane, which must satisfy the condition that $\vec{\mathbf{r}} - \vec{\mathbf{r}}_0$ is perpendicular to $\vec{\mathbf{n}}$:



$$\begin{aligned} 0 =& \vec{\mathbf{n}} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) \\ =& \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ 0 =& a(x - x_0) + b(y - y_0) + c(z - z_0)\rangle \end{aligned} \tag{10}$$

Pushing the equation above one step further, let's expand the products and add up all the constant terms calling their sum k:

$$\begin{aligned} ax_0 + by_0 + cz_0 =& ax + by + cz \\ k =& ax + by + cz \end{aligned} \tag{11}$$

So that if you see an equation for a plane such as:

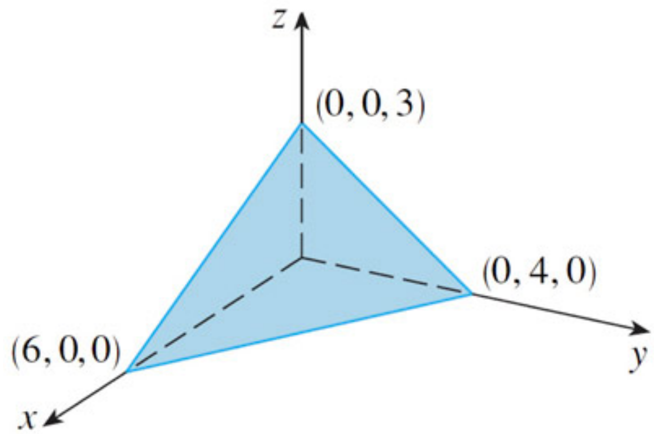
$$56 = 7x - 3y + 2z, \tag{12}$$

You can just read off the components of a normal vector to the plane as:

$$\vec{\mathbf{n}} = \langle 7, -3, 2 \rangle. \tag{13}$$

Example

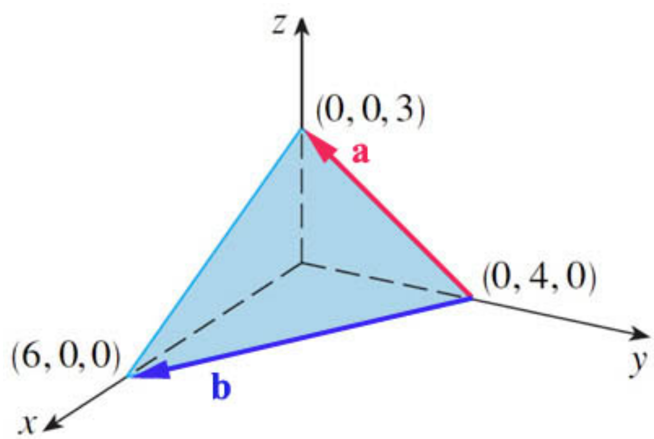
Example four from book, but using a cross product to find the surface normal...



First find the normal to the plane.
[Find two vectors that are in the plane, and then take their cross product.]

Then find the equation for the plane. [Take as $\vec{\mathbf{r}}_0$ any of the three points shown.]

Find the surface normal:
$\vec{\mathbf{a}} = \langle 0, -4, 3 \rangle$
$\vec{\mathbf{b}} = \langle 6, -4, 0 \rangle$
$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 12\,\hat{\mathbf{i}} + 18\,\hat{\mathbf{j}} + 24\,\hat{\mathbf{k}}$



Let's take for the normal vector...

$$\vec{\mathbf{n}} = 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}} \tag{14}$$

Picking $(0, 4, 0)$ as the point in the plane, then the prescription above is...

$$\begin{aligned} 0 =& a(x - x_0) + b(y - y_0) + c(z - z_0) \\ =& 2(x) + 3(y - 4) + 4(z) = 2x + 3y - 12 + 4z \end{aligned} \tag{15}$$

So, the equation for the plane can be written as:

$$2x + 3y + 4z = 12. \tag{16}$$

where a normal vector to the plane was $\vec{\mathbf{n}} = \langle 2, 3, 4 \rangle$.

Notice that when the equation for a plane is written in this form that you can **read off** the components of the normal vector from the coefficients of x, y, and z!

Using algebra, find the intersection of the planes $x + 2y + z = 4$ and $4x + 2y + 3z = 12$ in parametric form: For example, you could...

1. *solve the first equation for z (which depends on x and y),*
2. *substitute z into the second equation and solve it for y in terms of x, that is $y(x)$.*
3. *go back to your equation for $z(x, y)$. Substitute in your equation $y(x)$ to find z(x).*

Now you can express the coordinates of your line as $(x, y(x), z(x))$. This is a parametric expression for the line, in terms of a single parameter x.

Confirm by graphing the two planes and the line...

Image credits

vgm8383