Derivatives and integrals of vector functions [10.2]

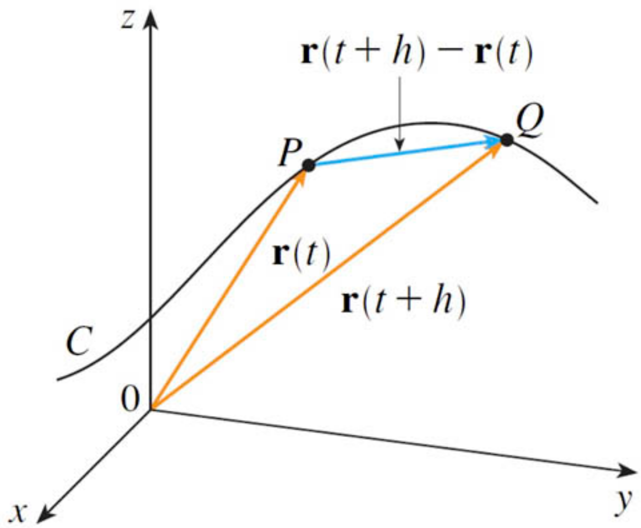
Derivatives

In Calc I, we defined the derivative as the limit of a "Difference Quotient":

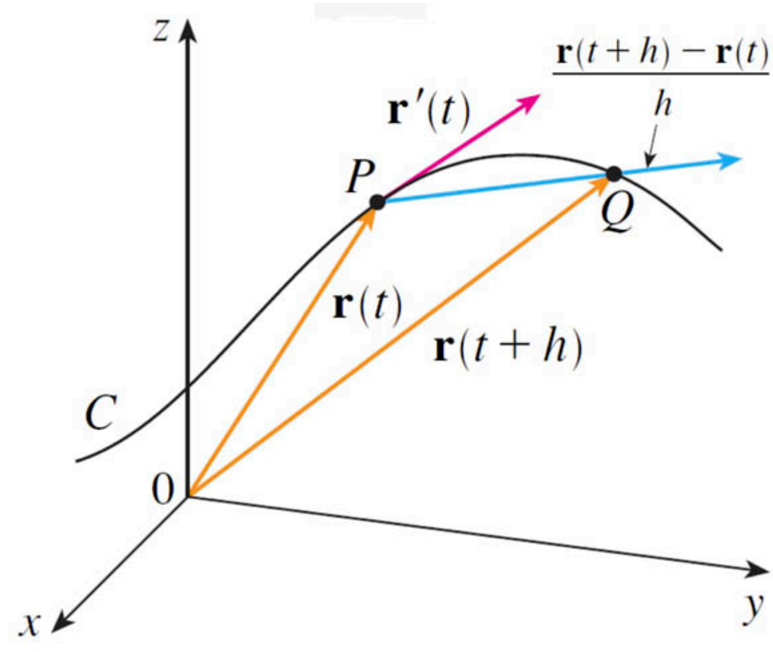
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \qquad (1)$$

For a vector function, $\vec{\mathbf{r}}(t)$, there's a similar difference quotient:

$$\frac{d\vec{\mathbf{r}}}{dt} \equiv \vec{\mathbf{r}}'(t) = \lim_{h\to 0} \frac{\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)}{h}. \qquad (2)$$



In the limit $h \to 0$, $\vec{\mathbf{r}}'(t)$ is a **vector** which is **tangent** to the curve at $\vec{\mathbf{r}}(t)$.



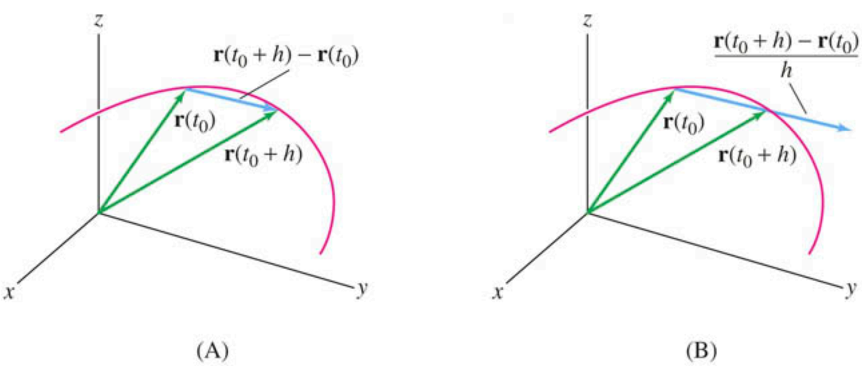


FIGURE 2 The difference quotient points in the direction of $\Delta\mathbf{r} = \mathbf{r}(t_0 + h) - \mathbf{r}(t_0)$.

Mess around with thisDifference Quotient (DQ) visualization (GeoGebra) to see these relationships.

Differentiation - components

A theorem: if $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$, where f, g, and h are differentiable functions, then

$$\vec{\mathbf{r}}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}} + h'(t)\,\hat{\mathbf{k}}. \quad (3)$$

Differentiation rules

3 Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Cusp?

A vector function $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$ might have a **cusp** if $\vec{\mathbf{r}}'(t) = 0$.

At right: $\vec{\mathbf{r}}(t) = \langle t^2 - 4t, e^{t-2} - t \rangle$, so $\vec{\mathbf{r}}'(t) = \langle 2t - 4, e^{t-2} - 1 \rangle$.

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ParametricPlot[
 {t^2 - 4 t, Exp[t - 2] - t},
 {t, 0, 4}, PlotStyle → Red]
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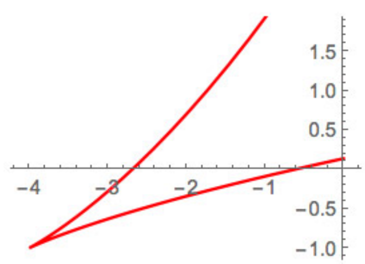
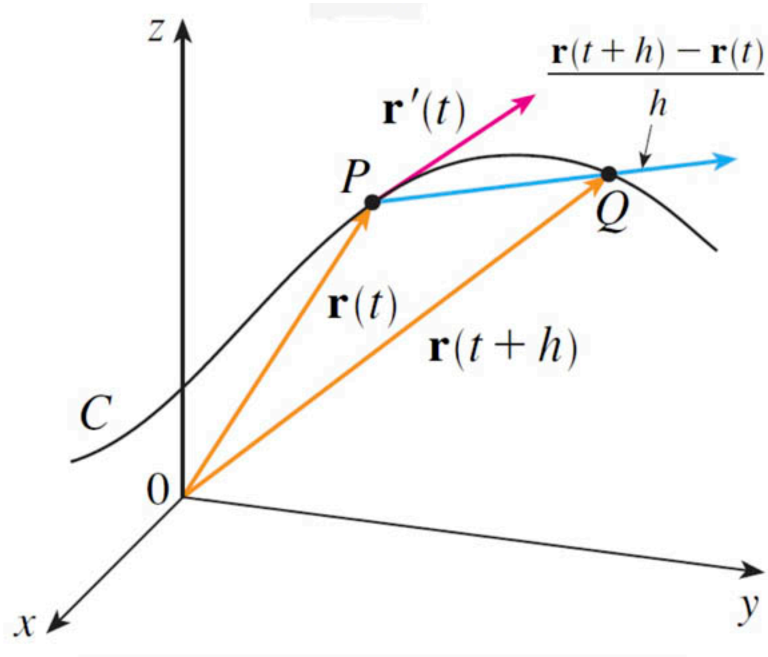


And... $\vec{\mathbf{r}}'(2) = \langle 0, 0 \rangle$.

More on tangent vectors



The line tangent to the curve at $\vec{\mathbf{r}}(t_0)$ is:

$$\vec{\mathbf{L}}(t) = \vec{\mathbf{r}}(t_0) + t\vec{\mathbf{r}}'(t_0). \tag{4}$$

The **unit tangent vector** is given by:

$$\hat{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}. \tag{5}$$

Theorem: If a vector function, $\vec{\mathbf{r}}(t)$, has constant length, then its derivative, $\vec{\mathbf{r}}'(t)$, is perpendicular to $\vec{\mathbf{r}}(t)$.

Proof:

1. We're assuming that the length of $|\vec{\mathbf{r}}(t)|$ is not changing. That is,

$$\frac{d}{dt}|\vec{\mathbf{r}}(t)| = 0. \tag{6}$$

2. Then, the length^2 is also unchanging:

$$\begin{aligned} 0 =& \frac{d}{dt}|\vec{\mathbf{r}}(t)|^2 \\ =& \frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \vec{\mathbf{r}}' \cdot \vec{\mathbf{r}} + \vec{\mathbf{r}} \cdot \vec{\mathbf{r}}' \\ 0 =& 2\vec{\mathbf{r}}' \cdot \vec{\mathbf{r}} \end{aligned} \tag{7}$$

3. Since the dot product of $\vec{\mathbf{r}}'$ and $\vec{\mathbf{r}}$ is zero, $\vec{\mathbf{r}}' \perp \vec{\mathbf{r}}$.

Example: $\vec{\mathbf{r}}(t) = \langle cos(t), sin(t) \rangle$. Taking the derivative of its component functions...

$$\vec{\mathbf{r}}'(t) = \langle -\sin(t), \cos(t) \rangle.$$

The magnitude of this derivative vector is

$$|\vec{\mathbf{r}}'(t)| = \sqrt{(-\sin t)^2 + \cos t)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1. \tag{8}$$

We see that the magnitude of the derivative is a constant (not changing with t).

See Circular motion (GeoGebra)

The unit tangent

- The Unit Tangent $\hat{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}$ has constant length 1.

- The only characteristic of $\hat{\mathbf{T}}(t)$ that changes as t changes is its direction.

- $|d\,\hat{\mathbf{T}}/dt|$ measures the **rate of change of the *direction*** of the unit tangent vector. ["units" are radians per t-unit].

Example: $\vec{\mathbf{r}}(t) = \langle t^3, t^6 \rangle$

Integrals

$$\begin{aligned}\int_a^b \vec{\mathbf{r}}(t)\,dt = &\int_a^b \langle f(t), g(t), h(t)\rangle\,dt \\ =&\left(\int_a^b f(t)\,dt\right)\,\hat{\mathbf{i}} + \left(\int_a^b g(t)\,dt\right)\,\hat{\mathbf{j}} + \left(\int_a^b h(t)\,dt\right) \\ \equiv&\vec{\mathbf{R}}(t).\end{aligned}$$

These are the components of $\vec{\mathbf{R}}(t)$, the anti-derivative of $\vec{\mathbf{r}}(t)$:

$$\int_a^b \vec{\mathbf{r}}(t)\,dt = \vec{\mathbf{R}}(t)\Big|_a^b = \vec{\mathbf{R}}(b) - \vec{\mathbf{R}}(a). \tag{10}$$

And $\vec{\mathbf{R}}'(t) = \vec{\mathbf{r}}(t)$.