# Arc length and curvature [10.3]

*The Nowitna river (Alaska) - How long is it?*

- Calculating arclength and curvature

- Arclength is independent of the choice of parameter used to describe a curve.

- Re-parameterizing a curve - using the **arclength as the parameter**.

- Geometric definition of curvature.

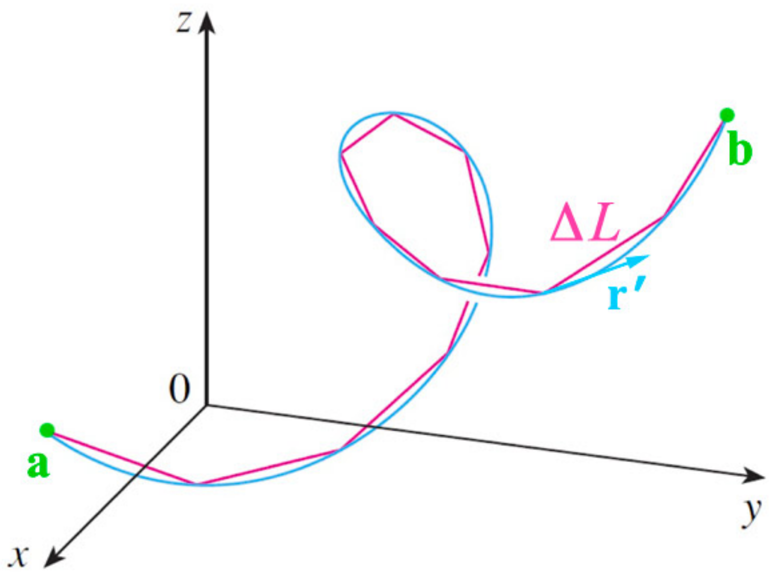
- The TNB frame (Tangent - Normal - Binormal unit vectors).

## Arclength

$\vec{\mathbf{r}}(t)$ is a vector function.

$$\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle; \;\; t_i \leq t \leq t_f.$$

where $\vec{\mathbf{r}}(t_i) = \vec{\mathbf{a}}$ and $\vec{\mathbf{r}}(t_f) = \vec{\mathbf{b}}$ The tip of the **position vector** $\vec{\mathbf{r}}(t)$ traces a curve (trajectory) in space which looks schematically like:




- The **arclength** is the distance *along the curve* from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$.

- Arclength is a **scalar**.

- It is **not the distance** "as the crow flows" of a vector starting at $\vec{\mathbf{a}}$ and terminating at $\vec{\mathbf{b}}$.

  The arclength from **a** to **b** is approximately equal to the sum of the pink segments:

$$\text{arclength} \equiv L \approx \sum \Delta L.$$

The derivative $\vec{\mathbf{r}}'(t)$ is a vector tangent to the curve. Its magnitude is the "speed" of a particle moving in time, $t$, according to $\vec{\mathbf{r}}(t)$.

Since distance = speed (times) time, we can approximate the distance $\Delta L$ moved in a time interval $\Delta t$ as

$$\Delta L \approx \left|\vec{\mathbf{r}}'\right| \Delta t$$

So, the arclength is approximately

$$L \approx \sum |\vec{\mathbf{r}}'(t)| \Delta t.$$

In the limit of ever smaller $\Delta t \rightarrow dt$, the sum approaches the integral...

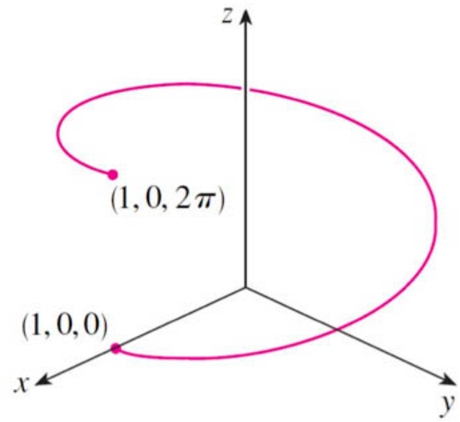
$$\begin{aligned} L &= \int_a^b |\vec{\mathbf{r}}'(t)|\, dt \\ &= \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt \\ &= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt \end{aligned} \tag{1}$$

## Example

Find the length of the arc of the helix that obeys the equation

$$\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t \rangle$$

from the point $(1,0,0)$ to $(1,0,2\pi)$.




It looks like

$$0 \leq t \leq 2\pi, \tag{2}$$

the derivative is

$$\vec{\mathbf{r}}'(t) = \langle -\sin t, \cos t, 1 \rangle,$$

and so...

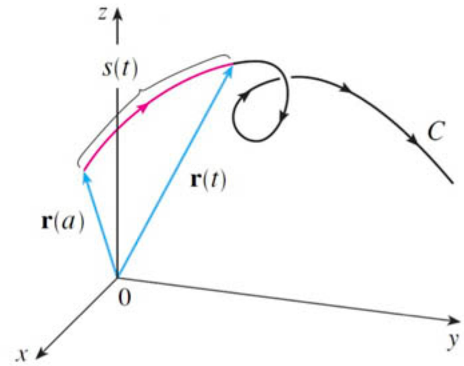
$$|\vec{\mathbf{r}}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}.$$

Substituting into the arclength expression

$$L = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

## Arclength distance function

The arclength distance function $s(t)$, where $s(0) = 0$,




$$s(t) = \int_0^t |\vec{\mathbf{r}}'(t)|\, dt = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \quad (3$$

.

Taking the derivative w.r.t. $t$ of both sides, this means that

$$\frac{ds}{dt} = |\vec{\mathbf{r}}'(t)|. \tag{4}$$

## Parameterizations

[Sorry, I can't bear to write "parametrization" as Stewart does.]

- The same curve can be represented in more than one way.
- E.g.

$$\vec{\mathbf{r}}_1(t) = \langle t, t^2, t^3 \rangle; \;\; 1 \le t \le 2$$

and

$$\vec{\mathbf{r}}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle; \;\; 0 \leq u \leq \ln 2 \tag{5}$$

*The two parametric curves aboves are plotted here (GeoGebra) (one as a point, and one as the tip of a position vector). Discuss with a partner:*

1. *Moving the $t$ and $u$ sliders back and forth… does it seem like the two functions trace out the same trajectory? Change the perspective a few times to make sure…*

2. *Next, animate the two functions (press the little play button besided each slider) and watch the two functions. Describe how the motions of the two functions differ.*

## Re-parameterization

It may be useful (and is certainly beautiful) to re-parameterize a curve in terms of the **arclength** along a curve

- is characteristic of the curve itself,

- does not depend on the choice of parameter used to describe the curve,

- and does not depend on the coordinate system.

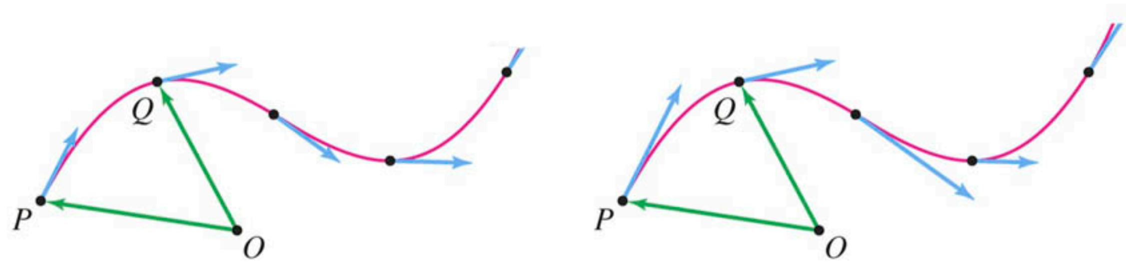
We could use the arclength ("distance along the curve") as the parameter to describe a curve (instead of the time, or angle, or some other parameter).

**Example** -- $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t \rangle$:Reparametrize with respect to arc length, beginning at (1,0,0) in direction of increasing $t$.

- $$|\vec{\mathbf{r}}'(t)| = \sqrt{f'^2 + g'^2 + h'^2} = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}. \quad ($$
- $$s(t) = \int_0^t |\vec{\mathbf{r}}'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}t. \qquad (7)$$
- Solve for $t$: $t = s/\sqrt{2}$,
- and substitute back into original expression:

$$\vec{\mathbf{r}}(t(s)) = \cos(s/\sqrt{2})\,\hat{\mathbf{i}} + \sin(s/\sqrt{2})\,\hat{\mathbf{j}} + s/\sqrt{2}\,\hat{\mathbf{k}}. \qquad (8)$$

This depends on being able (easily or otherwise) to take $s(t)$ and invert it to $t(s)$.




(A) An arc length parametrization (all tangent vectors have length 1)

(B) Not an arc length parametrization (tangent vectors' lengths vary)

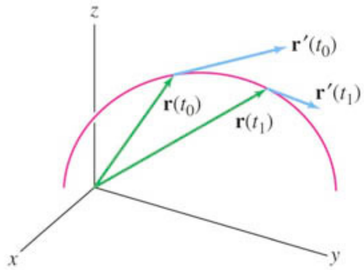
This is a technique used in General Relativity: We talk about the "proper time" measured along the the space-time trajectory of a particle as a quantity that all observers agree on.

## Curvature

The question: How to describe "curvature" in a useful way?

The curvature should be...

- a scalar number which should be
- large when the path is changing direction "quickly",
- small when the path is more nearly straight.




The velocity vector $\vec{\mathbf{r}}'(t)$ is tangent to the curve at each position on the curve. So, as the direction of a curve changes,the velocity vector will also change. So our first attempt to quantify the idea of "curvature" might be to look at at the *change of the velocity vector*, that is, the **acceleration**:

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{r}}'(t)}{dt} \tag{9}$$

Can you think of a situation in which the acceleration is not zero, even though the curvature of the path is zero?

The unit tangent

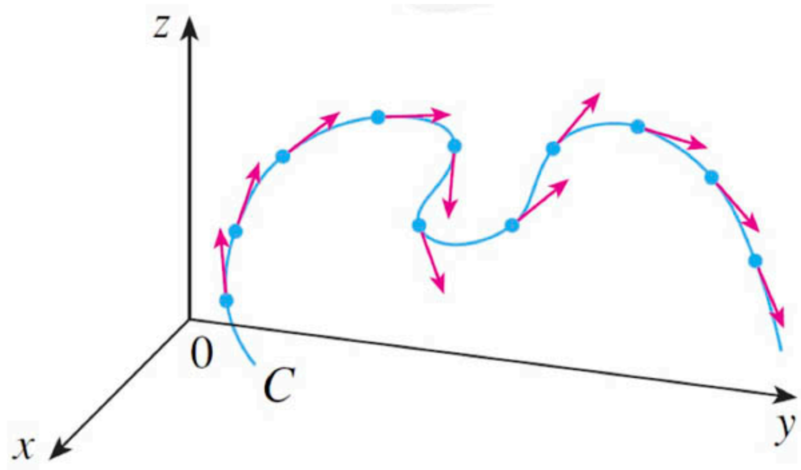
$$\hat{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'}{|\vec{\mathbf{r}}'|} \tag{10}$$

is a vector quantity that is always the same length, no matter what the velocity $\vec{\mathbf{r}}'$ is. So, what about

$$\frac{d\,\hat{\mathbf{T}}(t)}{dt}? \tag{11}$$

What's the problem with that?

In this picture, the unit tangent vectors are evenly spaced along the curve:




Does this suggest a useful way to think about curvature?

...We should look at how $\hat{\mathbf{T}}$ changes with respect to changes in the *arclength*, $s$, (instead of w.r.t time $t$):

The **curvature**, $\kappa$, is

$$\kappa = \frac{d\,\hat{\mathbf{T}}}{ds}, \tag{12}$$

the rate of change of unit tanget vector direction with respect to arclength.

By the chain rule, this is

$$\kappa = \left|\frac{d\,\hat{\mathbf{T}}}{dt}\right| \frac{dt}{ds} = \frac{\left|\frac{d\,\hat{\mathbf{T}}}{dt}\right|}{\frac{ds}{dt}} == \frac{|d\,\hat{\mathbf{T}}(t)/dt|}{|\vec{\mathbf{r}}'(t)|}. \tag{13}$$

## Curvature of a circle

It seems like a circle ought to have a **constant** curvature. Let's see...

Circle of radius $a$:

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + a\sin t\,\hat{\mathbf{j}}. \tag{14}$$

$$\vec{\mathbf{r}}'(t) = -a\sin t\,\hat{\mathbf{i}} + a\cos t\,\hat{\mathbf{j}}. \tag{15}$$

$$|\vec{\mathbf{r}}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a. \tag{16}$$

$$\hat{\mathbf{T}}(t) = \vec{\mathbf{r}}'(t)/|\vec{\mathbf{r}}'(t)| = \vec{\mathbf{r}}'(t)/a = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}. \tag{17}$$

$$\hat{\mathbf{T}}'(t) = -\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}}. \tag{18}$$

$$|\,\hat{\mathbf{T}}'\,| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1. \tag{19}$$

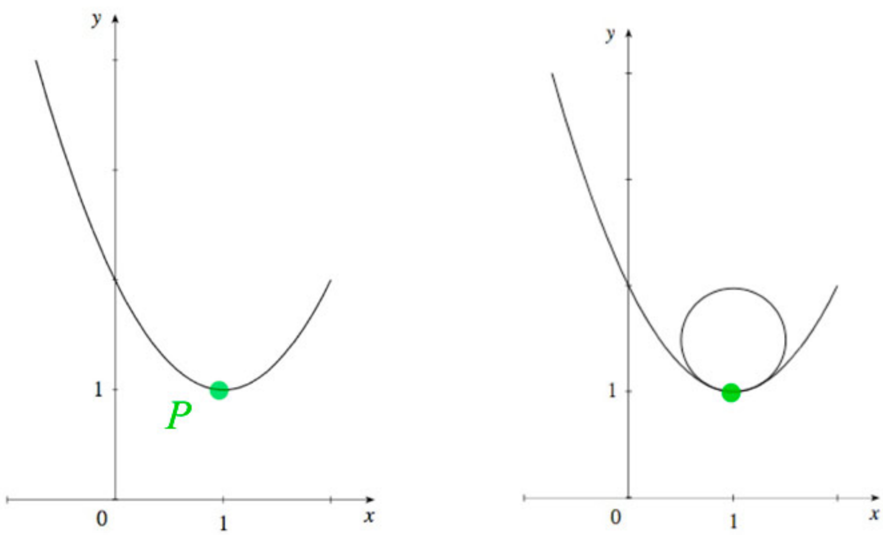
$$\kappa = \frac{|\,\hat{\mathbf{T}}'(t)|}{|\vec{\mathbf{r}}'(t)|} = \frac{1}{a}. \tag{20}$$

- The curvature $1/a$ is constant, independent of $t$.
- **Small circles** have a **large curvature**, and
- **large circles** have a **small curvature**.

- A **straight line** has a curvature of 0, because there is no change of the unit tangent vector.

This suggests a **geometric definition** of curvature...

## Osculating circle




The circle with the same curvature (that is, radius=$1/\kappa$) as $\vec{\mathbf{r}}(t)$ at a point $P$ on the curve.

## Another way to calculate curvature

We had:

$$\kappa = \left|\frac{d\,\hat{\mathbf{T}}}{dt}\right|\frac{dt}{ds} = \frac{|d\,\hat{\mathbf{T}}(t)/dt|}{|\vec{\mathbf{r}}'(t)|} = \frac{\left|\frac{d}{dt}\left[\frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}\right]\right|}{|\vec{\mathbf{r}}'(t)|}. \tag{21}$$

**Theorem**: The curvature of the curve given by $\vec{\mathbf{r}}(t)$ is given by:

$$\kappa(t) = \frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|^3}. \tag{22}$$

### Example 4 - group work

...uses the formula above to calculate the curvature of the 'twisted cubic' $\vec{\mathbf{r}}(t) = \langle t, t^2, t^3 \rangle$:

$$\kappa(t) = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}} \tag{23}$$

The twisted cubic...[plot the twisted cubic.]

- Would you expect the curvature function to be symmetric or not about $t = 0$ which corresponds to $(0, 0, 0)$?

- Would you expect there to be a maximum, minimum, or neither at $t = 0$?

[Graph $\kappa(t)$ to see...]

## Unit tangent, unit normal, binormal

Several quantities, defined in terms of derivatives of $\vec{\mathbf{r}}(t)$:

The unit tangent vector is

$$\hat{\mathbf{T}}(t) \equiv \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}. \tag{24}$$

Under what circumstances is $\hat{\mathbf{T}}'(t) = 0$?

(That is to say, $\hat{\mathbf{T}}' \equiv \frac{d}{dt}\hat{\mathbf{T}}$ is not necessarily a *unit* vector.)

But when $|\hat{\mathbf{T}}'(t)| \neq 0$...

- $\hat{\mathbf{T}}(t)$ is a vector whose *length* (=1) is not changing with time.
- Therefore $\hat{\mathbf{T}}'(t)$ must be at right angles to ("normal to") $\hat{\mathbf{T}}$.

The unit **normal** vector is

$$\hat{\mathbf{N}}(t) \equiv \frac{\hat{\mathbf{T}}'(t)}{|\hat{\mathbf{T}}'(t)|}. \tag{25}$$

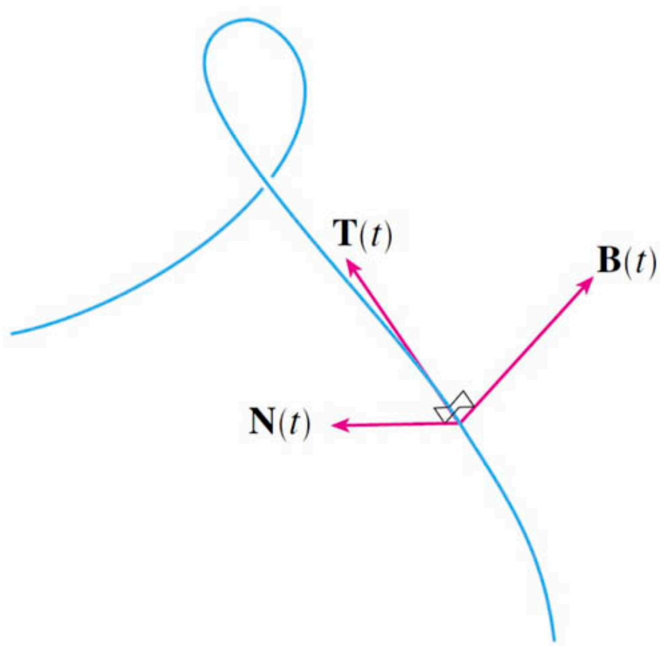
$\hat{\mathbf{N}}$ points towards the center of the osculating circle for $\vec{\mathbf{r}}(t)$.

What's the magnitude of the cross product of two unit vectors, which are at right angles?

The **binormal** vector is

$$\vec{\mathbf{B}}(t) = \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}(t). \tag{26}$$

We may as well write it as $\hat{\mathbf{B}}(t)$ because it *must* have unit length by this definition.




## Example

For $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}$: Find $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ at $t = 3\pi/2$. Sketch these on a plot of $\vec{\mathbf{r}}(t)$.

If $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 3t\,\mathbf{k}$, find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$. Sketch $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ when $t = \frac{3\pi}{2}$.