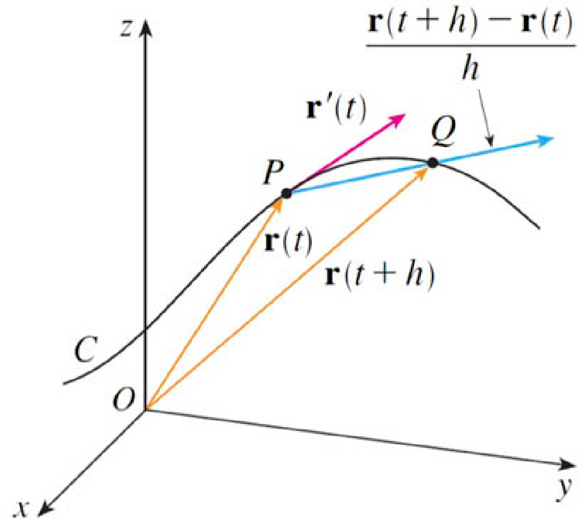
Motion in space [10.4]

If $\vec{\mathbf{r}}(t)$ represents the vector position of a point particle in space, as a function of time t, then the velocity vector is...



$$\vec{\mathbf{v}}(t) \equiv \lim_{h\to 0} \frac{\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)}{h} = \vec{\mathbf{r}}'(t) \equiv \dot{\vec{\mathbf{r}}}(t). \tag{1}$$

Speed

$$|\vec{\mathbf{v}}(t)| \equiv |\vec{\mathbf{r}}'(t)| = \frac{ds}{dt} \equiv v(t). \tag{2}$$

So the scalar speed $v(t)$ only conveys the *magnitude* of the velocity vector, nothing about its direction.

Acceleration

$$\vec{\mathbf{a}}(t) \equiv \vec{\mathbf{v}}'(t) = \vec{\mathbf{r}}''(t) \equiv \ddot{\vec{\mathbf{r}}}(t). \tag{3}$$

And we can also write acceleration as the limit of the change in velocity per unit time

$$\vec{\mathbf{a}}(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)}{\Delta t}. \tag{4}$$

Integrating...

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}(t_0) + \int_{t_0}^{t} \vec{\mathbf{a}}(u)\, du. \tag{5}$$

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(t_0) + \int_{t_0}^{t} \vec{\mathbf{v}}(u)\, du. \tag{6}$$

Projectile motion

Acceleration due to gravity is constant, and always *down*.

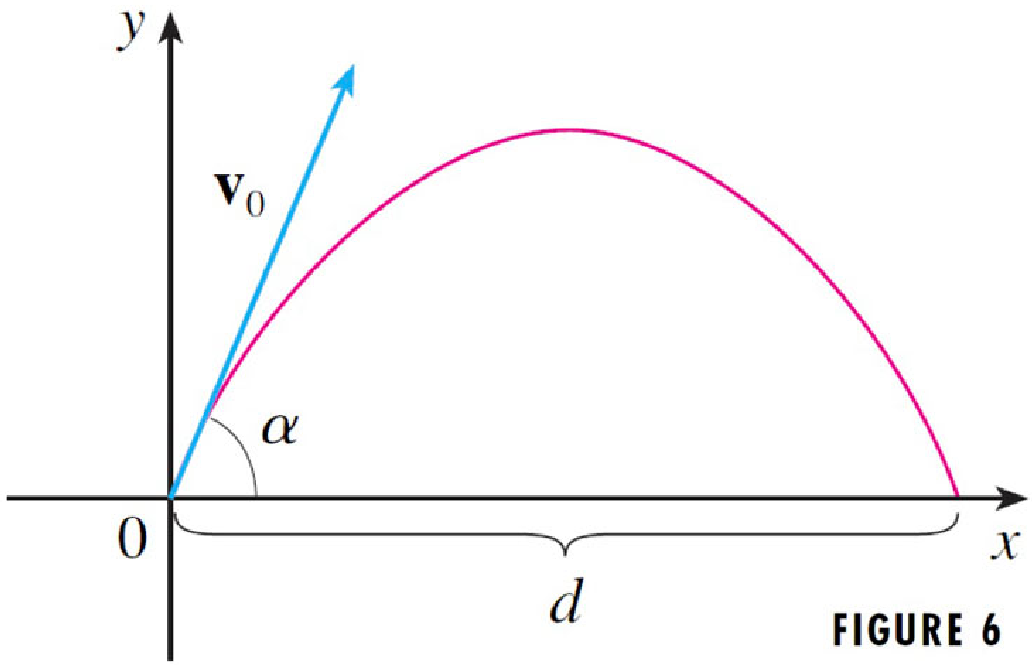


FIGURE 6

Newton's law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$: The only force acting on the projectile is gravity. The gravitational force is the object's weight, mg (where $g = 9.8 \text{ m/s}^2$).

$$\vec{\mathbf{F}} = -mg\,\hat{\mathbf{j}} = m(a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}}. \tag{7}$$

In the vertical direction ($\hat{\mathbf{j}}$ component) this becomes:

$$-g = a_y(t) = v_y'(t) \tag{8}$$

Integrating as above and saying that $t = 0$ is the time of launch (so that $v_y(0) = v_0 \sin\alpha$ and $v_x(0) = v_0 \cos\alpha$) we get:

$$v_y(t) = v_0 \sin\alpha - gt. \tag{9}$$

Integrating once more, as above, gives:

$$y(t) = (v_0 \sin\alpha)t - \frac{1}{2}gt^2. \tag{10}$$

Going through a similar process, the horizontal position is found to be:

$$x(t) = (v_0 \cos \alpha)t. \tag{11}$$

See the Catapult worksheet

Acceleration- components

Interesting things happen when we resolve acceleration into components parallel (tangential) and perpendicular (normal) to the velocity...

Since the unit tangent vector $\hat{\mathbf{T}} = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} = \frac{\vec{\mathbf{v}}}{v}$,

$$\vec{\mathbf{v}} = v\,\hat{\mathbf{T}}. \tag{12}$$

Differentiating both sides of this equation, using the product rule on the right, gives us

$$\vec{\mathbf{v}}' = \vec{\mathbf{a}} = v'\,\hat{\mathbf{T}} + v\,\hat{\mathbf{T}}'. \tag{13}$$

Now re-writing 2 relations from 10.3 in terms of velocity/speed: The curvature, κ,

$$\kappa = \frac{|\,\hat{\mathbf{T}}'|}{|\vec{\mathbf{r}}'|} = \frac{|\,\hat{\mathbf{T}}'|}{v},$$

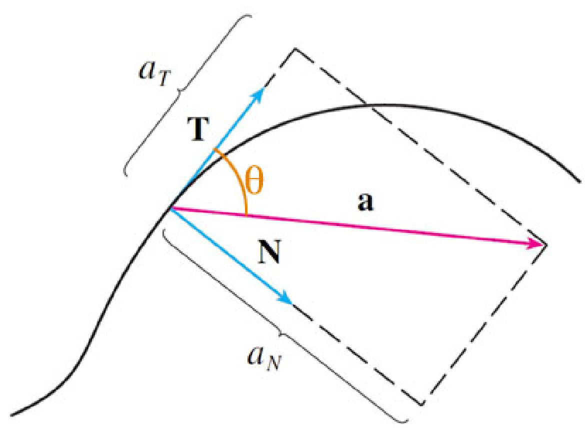
and the unit normal vector $\hat{\mathbf{N}} = \hat{\mathbf{T}}' / |\hat{\mathbf{T}}'|$,

$$\hat{\mathbf{T}}' = |\hat{\mathbf{T}}'|\,\hat{\mathbf{N}} = \kappa v\,\hat{\mathbf{N}}.$$

Substituting into 13,

$$\boxed{\vec{\mathbf{a}} = v'\,\hat{\mathbf{T}} + \kappa v^2\,\hat{\mathbf{N}}. \qquad (14)}$$

- No $\hat{\mathbf{B}}$ component (binormal) to the acceleration??!!
- This means that acceleration is always in the $\hat{\mathbf{T}}$-, $\hat{\mathbf{N}}$-plane.
- A change in speed, v' only affects the tangential component, a_T.



- The normal component of acceleration, a_N, is only sensitive to speed^2 and curvature, not the *change* in speed.
- Newton's law is: $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$.

- So, the "sideways force" that the rails/road exerts on a train/car in order to keep it on the track/road is the normal component of the force (also called the centripetal force):

$$F_N = ma_N = m\kappa v^2. \tag{15}$$

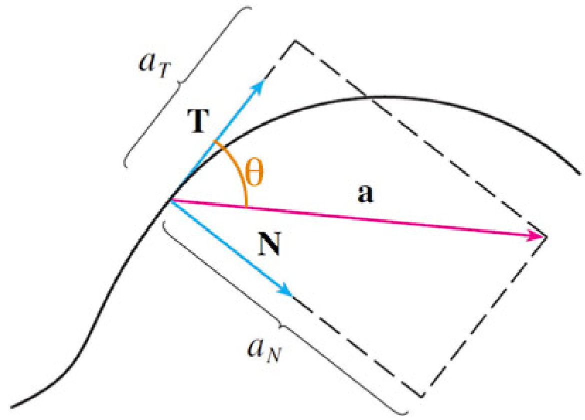
There is a maximum sideways force that rails can exert before a train "hops the rails"... 

"Twice the speed limit"...

If you double the speed, the sideways force needed to stay on the rails **quadruples**.

Components in terms of $\vec{\mathbf{a}}$

$$\vec{\mathbf{a}}(t) = a_T(t)\,\hat{\mathbf{T}}(t) + a_N(t)\,\hat{\mathbf{N}}(t). \tag{16}$$



[In what follows, I'll stop writing the explicit time dependence, but just remember that all the quantities we're dealing with are time dependent.]

The tangential component, a_T is the dot product of $\vec{\mathbf{a}}$ with a unit vector in the tangential direction, namely $\hat{\mathbf{T}}$:

$$\begin{aligned} a_T &= \vec{\mathbf{a}} \cdot \hat{\mathbf{T}} \\ &= \vec{\mathbf{a}} \cdot \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{v}. \end{aligned} \tag{17}$$

We can write the dot product as $\vec{\mathbf{a}} \cdot \hat{\mathbf{T}} = a\cos\theta$, where θ is the angle between $\vec{\mathbf{a}}$ and $\hat{\mathbf{T}}$.

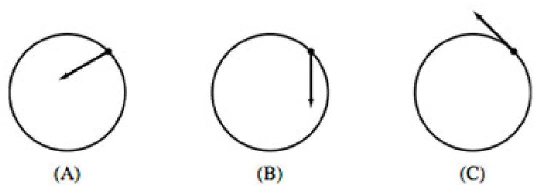
Since $\hat{\mathbf{T}} \perp \hat{\mathbf{N}}$, We could write

$$a_N = \vec{\mathbf{a}} \cdot \hat{\mathbf{N}} = a\sin\theta.$$

But $|\vec{\mathbf{a}} \times \hat{\mathbf{T}}| = a\sin\theta$ too, so

$$
\begin{aligned}
a_N =& |\vec{\mathbf{a}} \times \hat{\mathbf{T}}| \\
=& \left|\vec{\mathbf{a}} \times \frac{\vec{\mathbf{v}}}{v}\right| = \frac{|\vec{\mathbf{a}} \times \vec{\mathbf{v}}|}{v}.
\end{aligned} \tag{18}
$$

*[Test 1 question, 2015] A particle is moving counterclockwise around a circle. Which of the vectors in the figure below is **not** a possible acceleration vector? Explain. For the remaining two possible acceleration vectors, state whether the particle is speeding up or slowing down.*



Summary

Unit tangent vector

$$
\hat{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{v} \tag{19}
$$

Unit Normal vector

$$
\hat{\mathbf{N}} = \frac{\hat{\mathbf{T}}'}{|\hat{\mathbf{T}}'|} \tag{20}
$$

Decomposition of acceleration

$$\hat{\mathbf{a}}(t) = a_T(t)\,\hat{\mathbf{T}}(t) + a_N\,\hat{\mathbf{N}}(t) \tag{21}$$

Tangential component

$$a_T = v' = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{v} \tag{22}$$

Normal component

$$a_N = \kappa v^2 = \frac{|\vec{\mathbf{a}} \times \vec{\mathbf{v}}|}{v} \tag{23}$$

$$a_N\,\hat{\mathbf{N}} = \vec{\mathbf{a}} - a_T\,\hat{\mathbf{T}} = \vec{\mathbf{a}} - \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}\vec{\mathbf{v}} \tag{24}$$

For example

With this vector function for position:

$$\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}, \tag{25}$$

the velocity is

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = 2t\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}}. \tag{26}$$

The speed is

$$|\vec{\mathbf{v}}| = \sqrt{4t^2 + 4t^2 + 9t^4} = \sqrt{8t^2 + 9t^4}. \tag{27}$$

the acceleration is

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}''(t) = 2\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 6t\,\hat{\mathbf{k}}. \tag{28}$$

Tangential component of acceleration

$$a_T = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{v} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}. \tag{29}$$

Normal component of acceleration

$$a_N = \frac{|\vec{\mathbf{a}} \times \vec{\mathbf{v}}|}{v} = \frac{6\sqrt{2}t^2}{\sqrt{8t^2 + 9t^4}}, \tag{30}$$

using...

$$\begin{aligned} \vec{\mathbf{a}} \times \vec{\mathbf{v}} =& \left| \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 2 & 6t \\ 2t & 2t & 3t^2 \end{pmatrix} \right| \\ =& (6t^2 - 12t^2)\,\hat{\mathbf{i}} + (12t^2 - 6t^2)\,\hat{\mathbf{j}} + (4t - 4t)\,\hat{\mathbf{k}} \\ =& -6t^2\,\hat{\mathbf{i}} + 6t^2\,\hat{\mathbf{j}} \end{aligned} \tag{31}$$