Limits and Continuity [11.2]

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Sometimes where you end up depends on how you pursue your goals!

Limit definition

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \tag{1}$$

Meaning:

- As (x, y) approaches (a, b)...
- the limit of $f(x, y)$ is L...
- ***if*** we can make the value of $f(x, y)$ as close to L as we like...
- by taking (x, y) sufficiently close to the point (a, b), **but not equal to** (a, b)!

Examples

What happens as both x and y approach 0 for the function...

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}? \tag{2}$$

TABLE 1 Values of $f(x, y)$

x \ y	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

It appears that we can write:

$$\lim_{(x,y)\to(0,0)} f(x) = 1. \tag{3}$$

What happens as both x and y approach 0 for the function...

$$g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}? \tag{4}$$

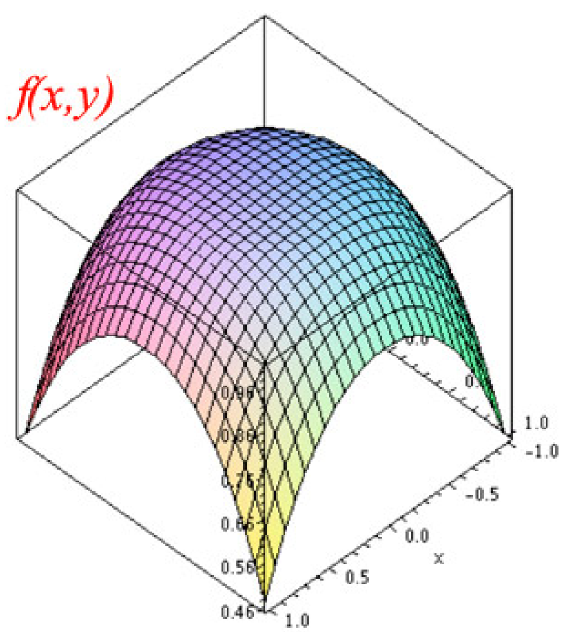
TABLE 2 Values of $g(x, y)$

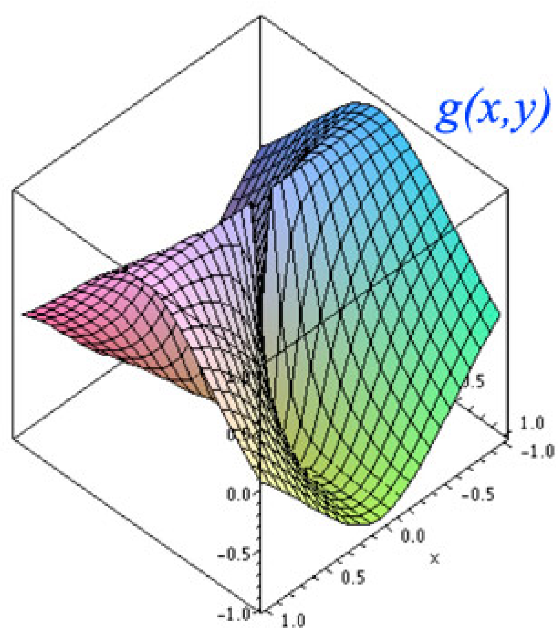
$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
-0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
-0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0	-1.000	-1.000	-1.000		-1.000	-1.000	-1.000
0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

It appears that:

$$\lim_{(x,y)\to(0,0)} g(x) \text{ does not exist.} \tag{5}$$

Graphs of the two function

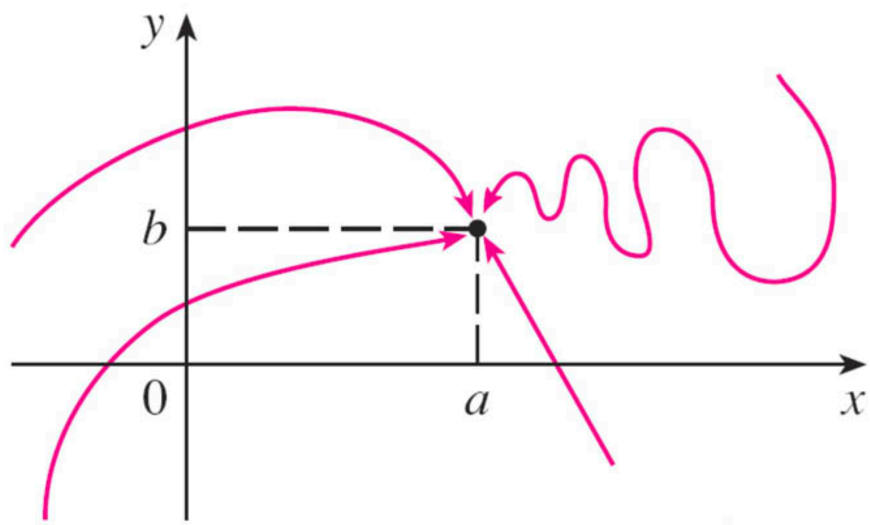




Direction of approach

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \tag{6}$$

means that the values of $f(x,y)$ approach the number L as the point (x,y) approaches the point (a,b) **along any path** that stays within the domain of f:



- Our definition says that the distance between $f(x,y)$ and L can be made arbitrarily small by making the distance from (x,y) to (a,b) sufficiently small (but not 0).
- The definition refers only to the distance between (x,y) and (a,b). It does not refer to the direction of approach.
- Therefore, if the limit exists, then $f(x,y)$ must approach the same limit no matter how (x,y) approaches (a,b).
- Thus, **if we can find two different paths of approach** along which the function has **different limits**, then $f(x,y)$ **has no limit** as (x,y) approaches (a,b).

For the function $g(x) = (x^2 - y^2)/(x^2 + y^2)$ the particular directions:

- Along the y-axis, $x = 0$, so $g(x,y) \to -y^2/y^2$, so apparently the limit when approaching the origin will be -1.
- But along the x-axis, $y = 0$, so $g(x,y) \to x^2/x^2$, so apparently the limit when approaching the origin will be 1.

...So we have two different paths of approach with different limits, and therefore ***the*** limit of the $g(x,y)$ is not unique near the origin, and does not exist.

Converse?

If we can show the same limit when approaching (a,b) from two different directions, does that mean that a function has a unique limit at (a,b)?

Consider the function $h(x,y)$

$$h(x, y) = \frac{xy}{x^2 + y^2}. \tag{7}$$

Does it have a limit at the origin $(0, 0)$ or not?

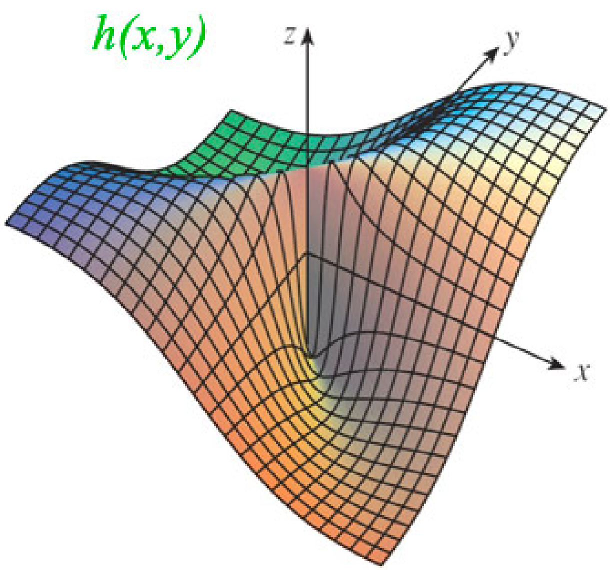
- **along the x-axis**: to get to the x-axis, $y \to 0$, and when this happens $f(x, y) \to 0/x^2 \to 0$.
- **along the y-axis**: to get to the y-axis, $x \to 0$, and when this happens $f(x, y) \to 0/y^2 \to 0$.

So, **same limit** from **two different directions**.

But approaching the origin along the line $x = y \equiv t$:

$$h(t, t) = t^2/(2t^2) \to 1/2. \tag{8}$$

So the limit along this line is 1/2, and *the* limit of h at $(0, 0)$ is not unique.

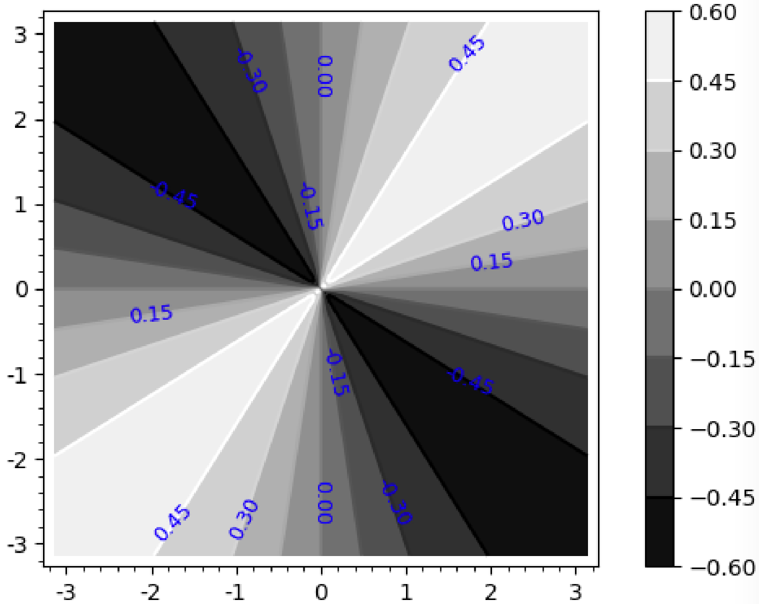


Graphically... What is the limit as you approach the origin along the line $x = -y$? View the graph on GeoGebra.

In [1]:

```
var('x y')
f(x,y)=(x*y)/(x^2+y^2)
contour_plot(f(x,y), (x,-pi,pi), (y,-
pi,pi),colorbar=True,labels=True)
```

Out[1]:



Continuity

A function f of two variables is called **continuous at** (a,b) if

$$\lim_{(x,y)\to(a,b)} = f(a,b). \tag{9}$$

We say f is **continuous on** D if f is continuous at every point (a,b) in D.

To do

- *Limits handout*

Image credits

[Theen Moy]