Which way to go?

The hiking question(s)

Which direction should I go...

- to go up most steeply?
- to go down most steeply?
- to keep moving around the mountain while staying at the same altitude?
- Also, how steep is the path I'm on right now?

Ummmmm... Actually our brains working with our eyes are pretty good at answering such questions *unconsciously* with little or not calculus training!

We will instead be trying to answer questions about **surfaces** for which we have a **mathematical description** (often a function). The surfaces will mostly be continuous and well behaved.

The slope...in any direction

For a function, $f(x)$, of one variable, $\frac{df}{dx}$ is ***the* slope** of the graph at x.

But on a mountain (=*surface*), **slope depends on the direction** of motion. Or we could say, it depends on the *path* taken.

Partial derivatives [11.3]

If f is a function of two variables, then its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \tag{1}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \tag{2}$$

In the first definition, y is constant and x varies. In the second one x is constant and y varies.

Notation...

$$f_x(x,y) \equiv f_x \equiv \frac{\partial f}{\partial x} \equiv \frac{\partial}{\partial x} f(x,y) \equiv \frac{\partial z}{\partial x} \equiv D_x f. \tag{3}$$

$$f_y(x,y) \equiv f_y \equiv \frac{\partial f}{\partial y} \equiv \frac{\partial}{\partial y} f(x,y) \equiv \frac{\partial z}{\partial y} \equiv D_y f. \tag{4}$$

with $z = f(x,y)$.

[I have seen f_1 or D_1 very rarely].

Rules for calculating partial derivatives

1. To find f_x, regard y as constant, and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as constant, and differentiate $f(x,y)$ with respect to y.

For example:

$$f(x,y) = 3xy^2 \tag{5}$$

$$f_x = \frac{\partial}{\partial x}(3y^2)x = 3y^2 \tag{6}$$

$$f_y = \frac{\partial}{\partial y}(3x)y^2 = (3x) * 2y = 6xy \tag{7}$$

In CoCalc (at right):

```
var('x y')
f(x,y)=x*y^2
show(f(x,y))
```

$$xy^2$$

```
show( diff(f(x,y),x) )
```

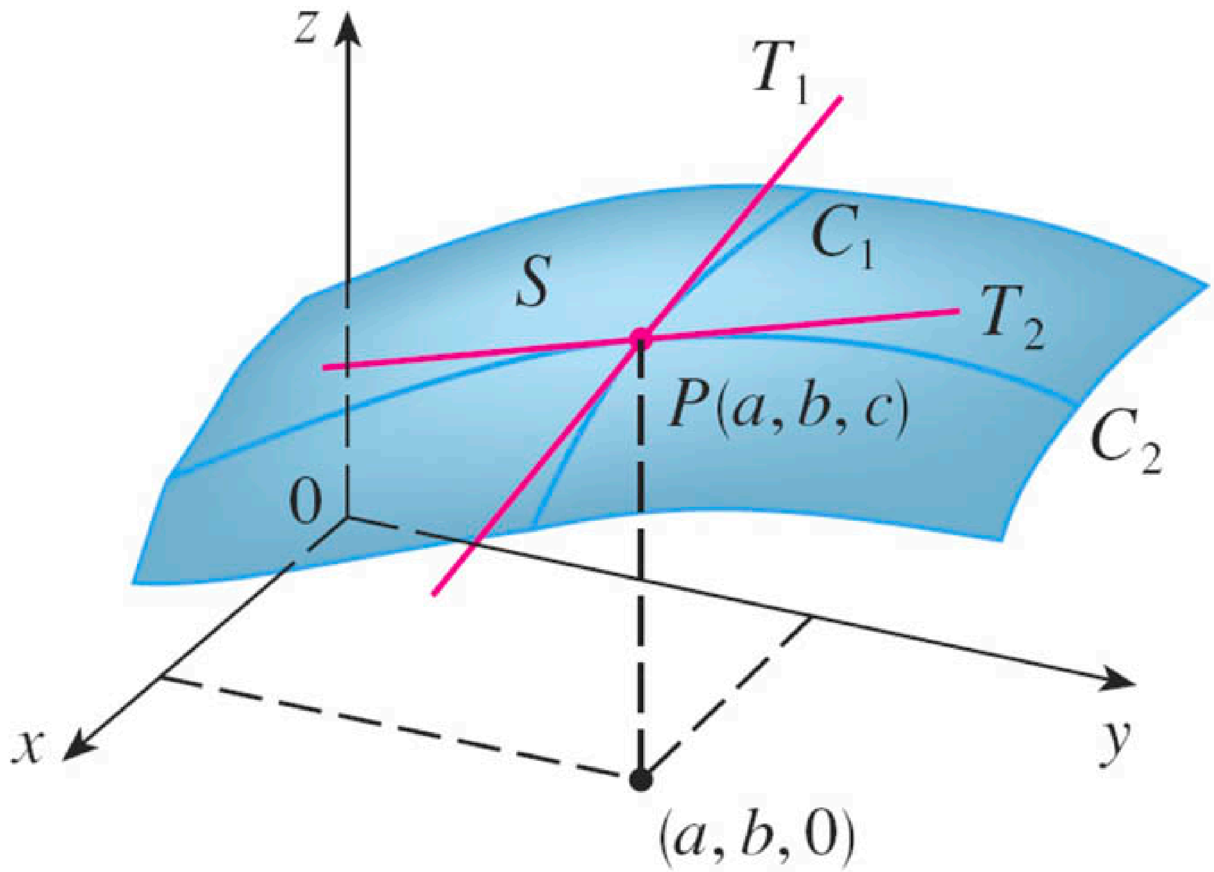
$$y^2$$

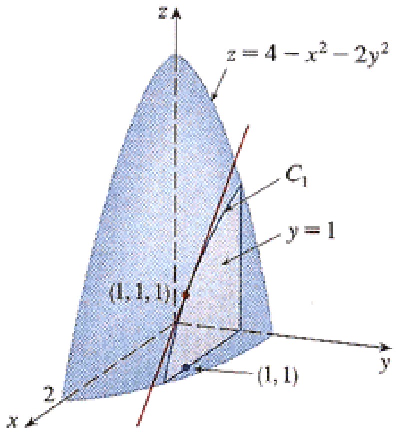
```
show(  diff(f(x,y),y))
```

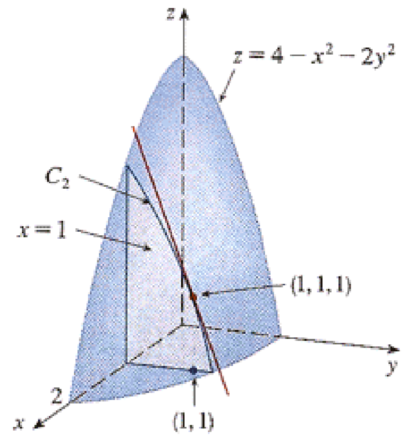
$$2\,xy$$

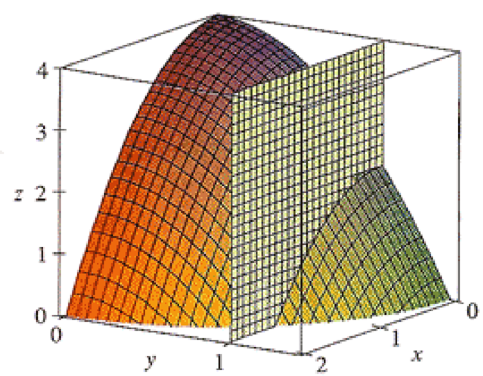
Mathematica: `D[f,x]`, `D[f,y]`

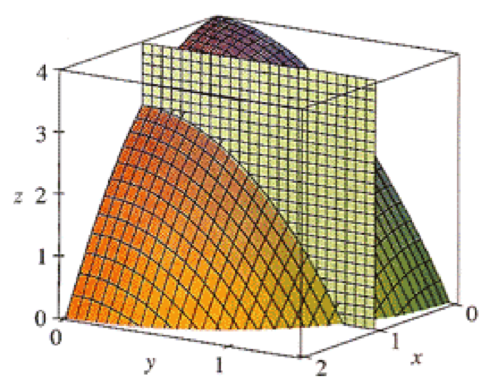
Visualizing

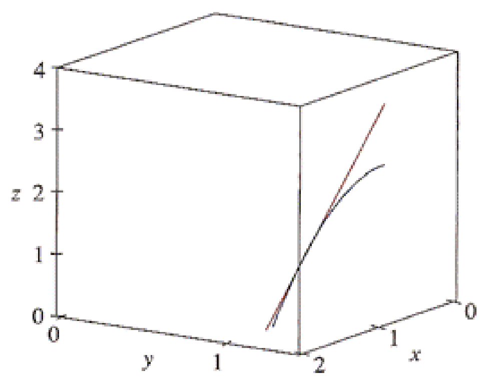

z
T_1
C_1
S
T_2
$P(a, b, c)$
C_2
0
x
y
$(a, b, 0)$

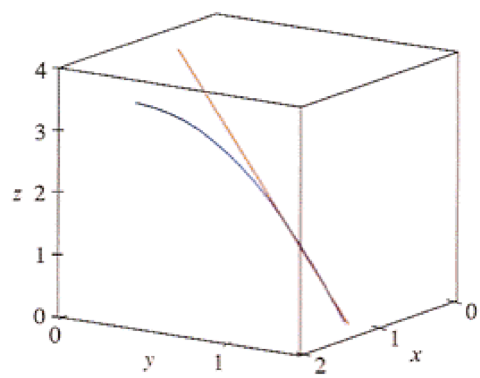












To do

- *Tabular data - Just 1-3.*
- *Graphical data - Just 1-3.*

Higher order derivatives

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}. \tag{8}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}. \tag{9}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}. \tag{10}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}. \tag{11}$$

Interpretation

f_x means...what?

f_{yy} means...what?

f_{yx} is the rate of change of the slope in the y direction, as you increase x. But is there a word like "concavity" or "slope" to describe what that means graphically? See this visualization of f_{yx}

Example

Consider...

$$f(x, y) = x^3 + x^2y^3 - 2y^2 \tag{12}$$

$$f_x = 3x^2 + 2xy^3; \quad f_y = 3y^2x^2 - 4y \tag{13}$$

$$f_{xy} = \frac{\partial f_x}{\partial y} = 6xy^2; \quad f_{yx} = \frac{\partial f_y}{\partial x} = 6xy^2 \tag{14}$$

$$f_{xx} = \frac{\partial f_x}{\partial x} = 6x + 2y^3; \quad f_{yy} = \frac{\partial f_y}{\partial y} = 6yx^2 - 4 \tag{15}$$

Clairaut's theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b). \tag{16}$$

Another way of saying this is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right). \tag{17}$$

See Clairaut's theorem and the meaning of f_{xy}.

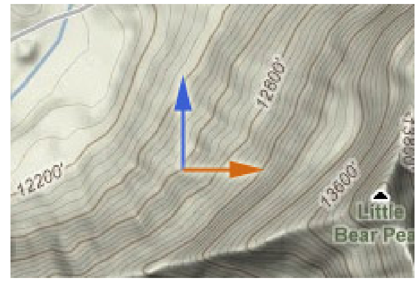
Differentiability

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \tag{18}$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to 0$.

In hiking terms



How does your altitude change (Δz)?

Well, it depends on how far east (Δx) and north (Δy) you move,

...and also on the the east-west slope (f_x) and the north-south slope. It's a pretty good approximation to say that your height will change by...

$$\Delta z \approx f_x \Delta x + f_y \Delta y. \tag{19}$$

This is, in principle, the resolution to at least one of the hiking questions:

- If you choose to step in an arbitrary direction, as specified by Δx and Δy,
- Then, knowing only two **local** characteristics of the surface, f_x and f_y,
- You can approximate your change in height.

Theorem: If the partial derivatives f_x and f_y exist near (a, b), and if they are continuous at (a, b), then f is differentiable at (a, b).

In hiking terms: The approach of using Eq (19) to calculate your change in height won't work on top of a sharp ridge, or at the edge of a cliff.

To Do

- *Mixed partials*

Image credits

Matthiew Lienhart