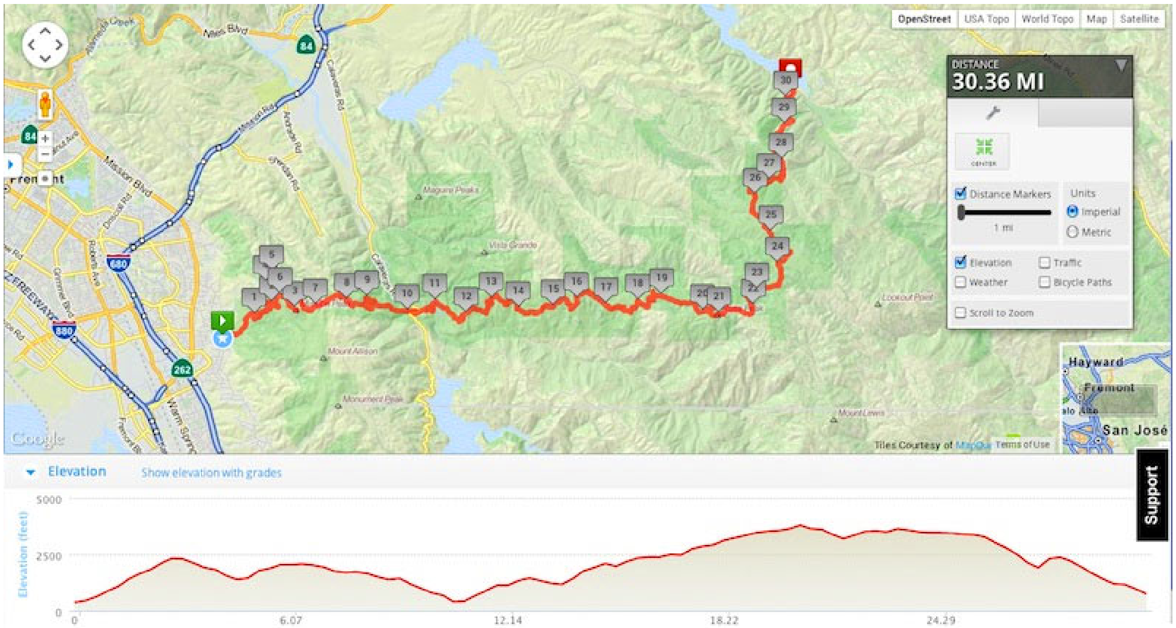
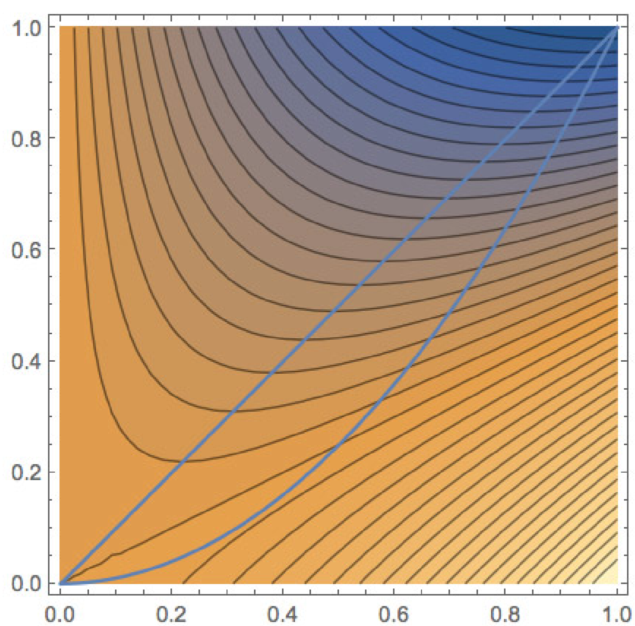
Lagrange multipliers



How to find peaks, or valleys, along a particular path?

1st approach



In the kitten problem, $T(x, y) = x^2 - 2xy$, you guessed that the highest temperature occurred on the bottom blue boundary, as close to the next-highest contour line outside as possible.

Now try another approach:

- *Parameterize the blue path,* $y = x^2$*, coming up with* $x(t)$ *and* $y(t)$*,*
- *Find* $z(t) = T(x, y) = T(x(t), y(t))$
- *Find critical points:* t *such that* $dz/dt = 0$*. If more than one, figure out the maxima.*
- *Use that* t *to find coordinates* $x(t)$ *and* $y(t)$*.*

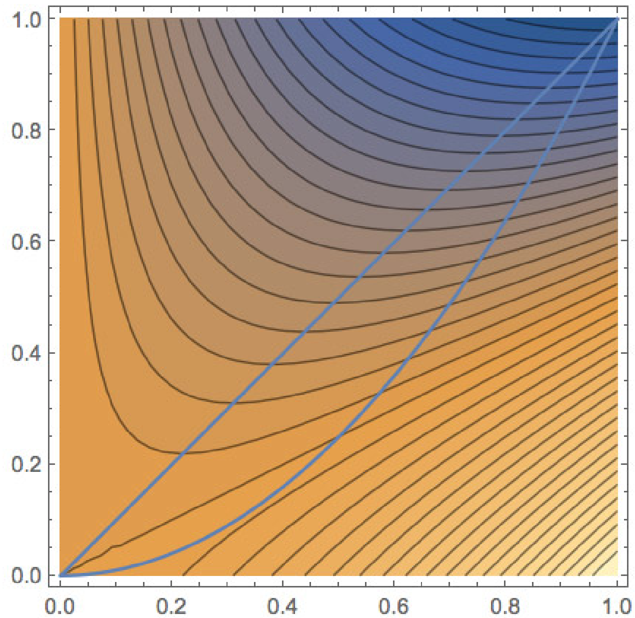
What are the coordinates of the point along the boundary with the highest temperature?

Now we'll consider another way...

The gradient and contour lines

$z = f(x, y)$ is the equation for a surface in three dimensions.

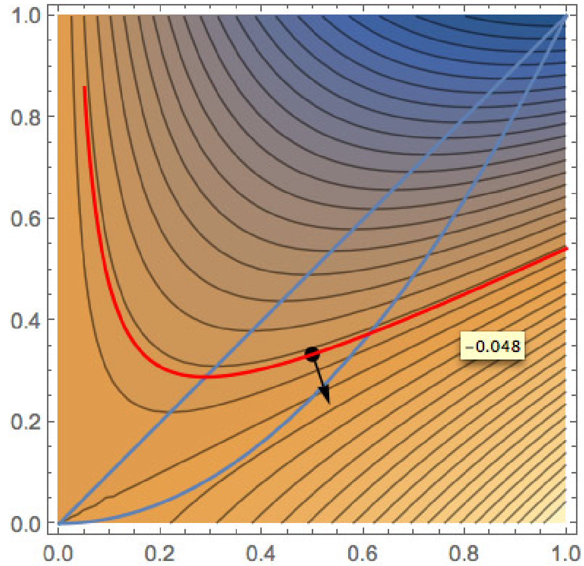
What we mean by a "contour line":



- $f(x, y) = k$ is a level curve or contour line.

- A contour line is a curve graphed in the xy plane. (Not in 3-dimensions.)

The gradient at a point (x_0, y_0) in the xy plane:



- $$\begin{aligned} \vec{\nabla} f(x_0, y_0) &= f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} \\ &\equiv \left.\frac{\partial f(x,y)}{\partial x}\right|^{(x_0,y_0)} \hat{\mathbf{i}} + \left.\frac{\partial f(x,y)}{\partial y}\right|^{(x_0,y_0)} \hat{\mathbf{j}} \end{aligned} \qquad (1)$$

- It points in the direction of maximum slope of the surface $z = f(x, y)$.

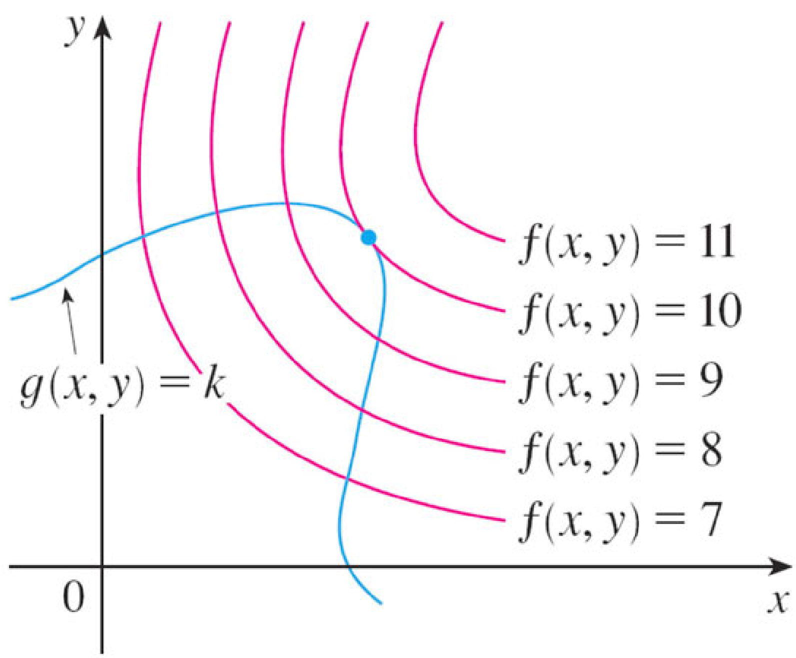
- But the gradient is a vector in the xy plane. (Not in 3-dimensions).

- The gradient at (x_0, y_0) is always perpendicular to the contour line $f(x, y) = k = f(x_0, y_0)$.

At every point along the 2-d curve $f(x, y) = k$, the gradient $\vec{\nabla} f$ is a **normal vector** of the curve.

Using gradients to find extreme points

Imagine that you are hiking along a path specified by the equation $g(x, y) = k$ through a landscape where the height is specified by $f(x, y)$. How do you find the highest point of your hike?



In mathematical terms:

- Try to find the **extreme values** of $f(x, y)$...
- Subject to the **constraint** $g(x, y) = k$.

We seek the the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$.

E.g. maximizing $f(x, y)$ means finding the contour $f(x, y) = c$ that $g(x, y)$ touches, where c has the largest possible value.

It looks like this happens when the desired contour of f is *tangent* to $g(x, y) = k$.

[If g is not tangent to a particular contour of f, then there exists a point on g which is higher, and one which is lower on either side of their intersection.]

Tangent means that **the normal vectors** to the 2-d curves $g = k$ and $f = c$ **are parallel** (one is the scalar multiple of the other).

We know $\vec{\nabla} f$ is always normal to any 2-d contour line of f. Ditto for $\vec{\nabla} g$ along the particular contour line $g = k$. Therefore...

At the location (x_0, y_0) where the two contours are tangent, it will also be the case that the normals are parallel, $\vec{\nabla} f \parallel \vec{\nabla} g$, which means:

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0) \tag{2}$$

where λ is some scalar.

The method of Lagrange multipliers

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

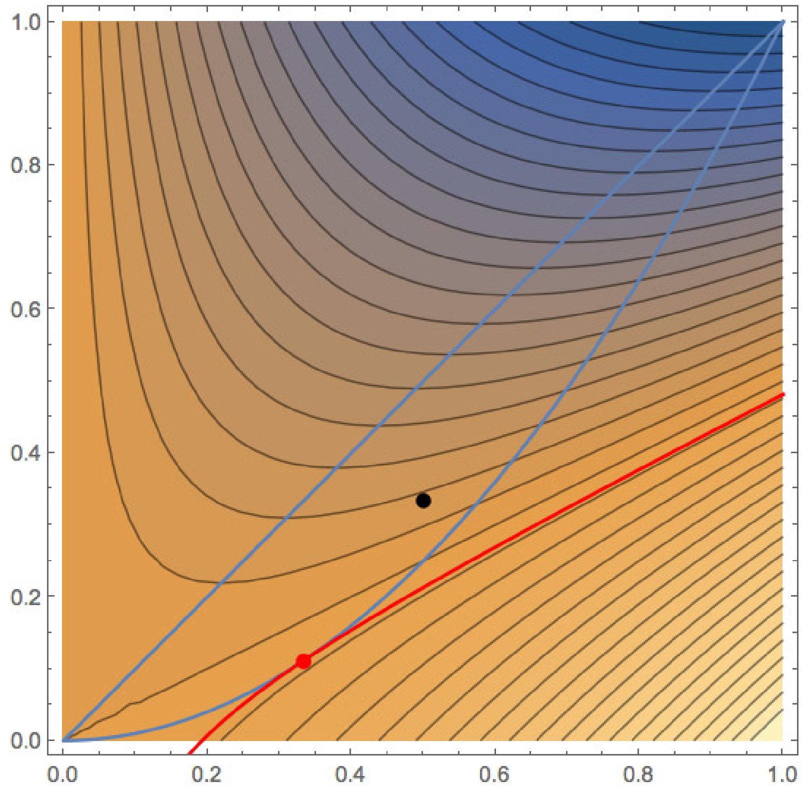
Kitten, re-re-visited

$$T(x, y) = x^2 - 2xy \tag{3}$$

The gradient of this function..

$$\vec{\nabla} T = T_x \,\hat{\mathbf{i}} + T_y \,\hat{\mathbf{j}} = (2x - 2y)\,\hat{\mathbf{i}} - 2x\,\hat{\mathbf{j}}. \tag{4}$$

We're interested in the the highest contour of T along the path $y = x^2$.



We need to express the path $y = x^2$ in the form of $g(x, y) = k$... How about

$$g(x, y) = x^2 - y = 0? \tag{5}$$

We can calculate the gradient of this function...

$$\vec{\nabla} g = 2x\,\hat{\mathbf{i}} - 1\,\hat{\mathbf{j}}. \tag{6}$$

The condition that the gradients are parallel,

$$
\begin{aligned}
\vec{\nabla} T =& \lambda \vec{\nabla} g \\
(2x - 2y)\,\hat{\mathbf{i}} - 2x\,\hat{\mathbf{j}} =& \lambda(2x\,\hat{\mathbf{i}} - 1\,\hat{\mathbf{j}}) \\
(2x - 2y)\,\hat{\mathbf{i}} - 2x\,\hat{\mathbf{j}} =& \lambda 2x\,\hat{\mathbf{i}} - \lambda\,\hat{\mathbf{j}}
\end{aligned} \tag{7}
$$

Two vectors are only equal if, separately, their x components are equal and their y components are equal.

The x-component are equal:

$$2x - 2y = \lambda 2x \tag{8}$$

and **the y-components** are equal:

$$-2x = -\lambda. \tag{9}$$

Using the two to eliminate the scalar λ:

$$2x - 2y = (2x)2x = 4x^2 \tag{10}$$

Taking into account our constraint equation, that $y = x^2$, leads to one equation for x:

$$
\begin{aligned}
2x - 2(x^2) =& 4x^2 \\
1 - x =& 2x
\end{aligned} \tag{11}
$$

The solution is $x = 1/3, \Rightarrow y = 1/9$. Hopefully the same location we found by parametric means?!

`Solve` ing simultaneous equations in Cocalc

```
In [13]: var('x y l')
         solve([
                 2*x-2*y == l*2*x,
                 -2*x == -l,
                 y == x^2
                 ], # 3 equations, in [...]
               x,y,l # variables to solve for
               )

Out[13]: [[x == (1/3), y == (1/9), l == (2/3)], [x == 0, y == 0, l == 0]]
```

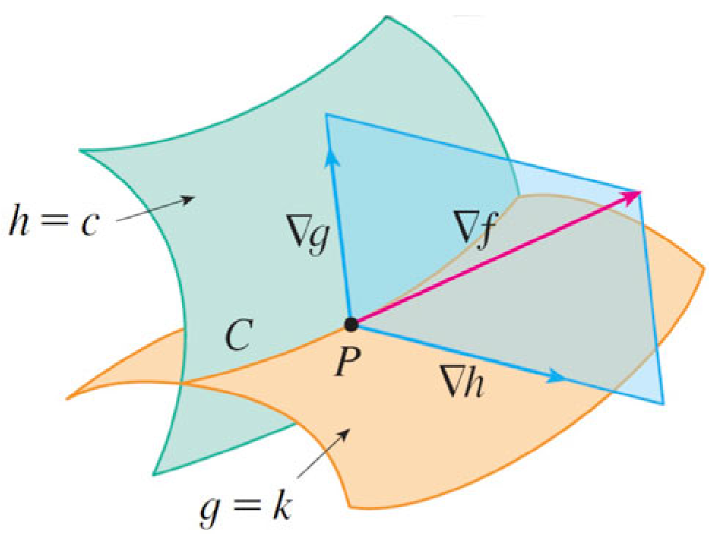
[or in *Mathematica*

Two constraints?

Suppose that we want to find...

- the maximum or minimum values of a function $f(x, y, z)$,
- subject to **two** constraints: $h(x, y, z) = c$ and $g(x, y, z) = k$.

Geometrically, we are looking for the extreme values of f when (x, y, z) lies on the curve of intersection, C, of the level surfaces



$h(x,y,z)=c$ and $g(x,y,z)=k$. Which can be expressed as...

$$\vec{\nabla} f(x_0,y_0,z_0)=\lambda\vec{\nabla} g(x_0,y_0,z_0)+\mu\vec{\nabla} h(x_0,y_0,z_0) \quad (12)$$

where λ and μ are scalars.

This amounts to solving these five equations...

$$f_x=\lambda g_x+\mu h_x \quad (13)$$

$$f_y=\lambda g_y+\mu h_y \quad (14)$$

$$f_z=\lambda g_z+\mu h_z \quad (15)$$

$$g(x,y,z)=k \quad (16)$$

$$h(x,y,z)=c \quad (17)$$

for the five unknown quantities λ, μ, x, y, and z.

ToDo

- *Lagrange Multiplier Problems: *Do* use CoCalc to solve the systems of equations.*