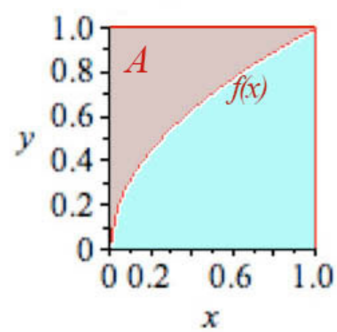
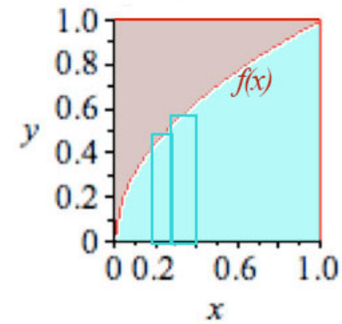
Area between curves

How to calculate the area in red above the function $f(x) = \sqrt{x}$?



Many ways!...

A single integral over x



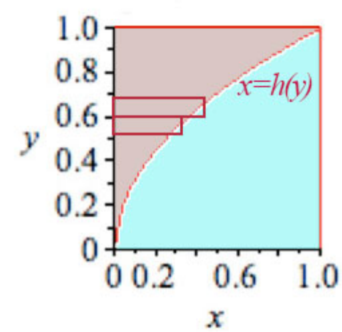
The area of the box is $1 \times 1 = 1$. Calculate the area below $f(x)$ with an integral, and subtract from 1:

$$\begin{aligned}\int_{x=0}^{1} \sqrt{x}\,dx &= \int_{0}^{1} x^{\frac{1}{2}}\,dx \\ &= \left.\frac{x^{\frac{3}{2}}}{3/2}\right|_{0}^{1} = \frac{2}{3}(1-0) = \frac{2}{3}.\end{aligned} \tag{1}$$

So our desired area is

$$A = 1 - 2/3 = 1/3. \tag{2}$$

A single integral over y



If $f(x) = y = \sqrt{x}$, then with a little algebra we can invert the function:

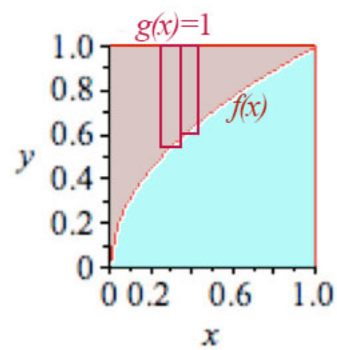
$$
\begin{aligned}
y =& \sqrt{x} \\
y^2 =& (\sqrt{x})^2 \\
y^2 =& x
\end{aligned} \tag{3}
$$

Express the red curve as the function $x = h(y) = y^2$ and integrate over y:

$$
\int_{y=0}^{1} y^2 \, dy = y^3/3\Big|_0^1 = 1/3 - 0 = 1/3. \tag{4}
$$

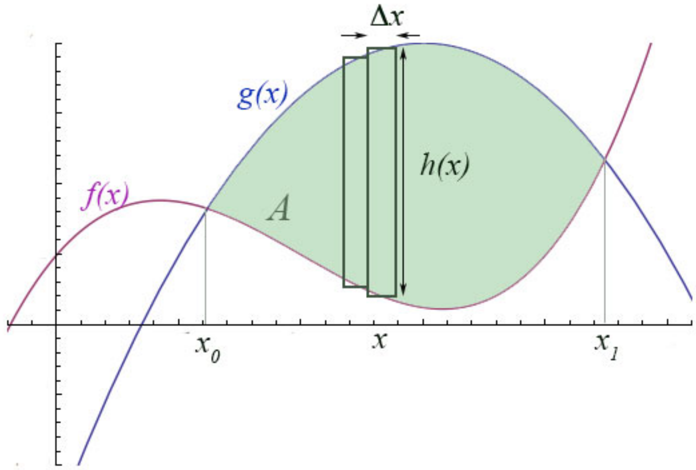
Ah, the same answer.

Area between curves



An integral over x is the Rieman sum of rectangles of a height which varies with x. That rectangle "height" could instead be given by the difference between the two functions shown

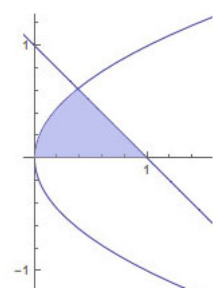
$$
\begin{aligned}
\int_{x=0}^{1} (g(x) - f(x)) \, dx =& \int_{x=0}^{1} (1 - \sqrt{x}) = \int_0^1 dx - \int_0^1 \sqrt{x} \, dx \\
=& x\Big|_0^1 - 2/3 = 1 - 2/3 = 1/3.
\end{aligned}
$$



To Do - Area Between Curves

1. *In* `Lab 06`*, Exercise 1, You'll plot the curves* $f(x) = \ln[x]$ *and* $g(x) = (x-1)^2$ *and find the area between the points of intersection of the functions.*

2. *In Exercise 2, you'll use parametric plotting to plot* $x = y^2$ *and* $x = 1 - y$*. Find the area of the region shown, between the two curves and* ***above the*** x***-axis****. Do this, by using* y *as your integration variable (instead of* x*).*

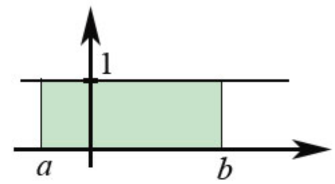


Integral of 1 as width of interval

This integral:

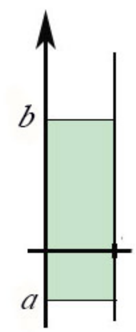
$$\int_a^b 1\,dx = \int_a^b 1\,dx = \int_a^b 1\,dx = x\big|_a^b = b - a \tag{6}$$

is a very contorted way of writing the **width of the interval** from a to b.



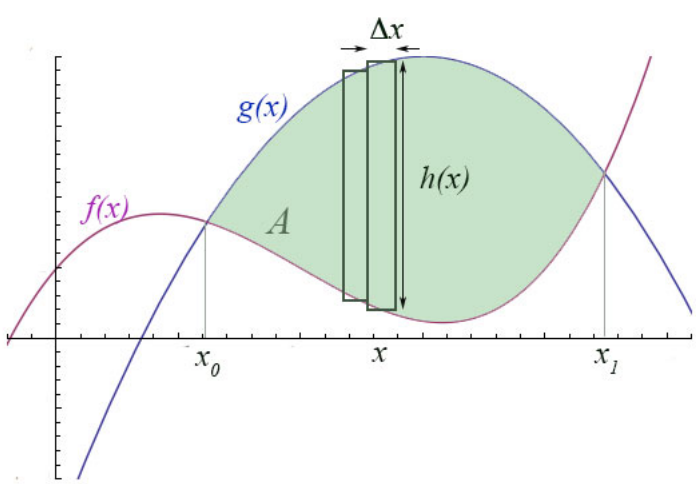
Or, turning the picture by 90 degrees, the vertical distance from a to b is

$$\int_a^b 1\, dy = b - a. \tag{7}$$



In the picture of the area between two curves, we could write the height $h(x) = g(x) - f(x)$ of the rectangle at x as

$$h(x) = \int_{y=f(x)}^{g(x)} dy. \tag{8}$$



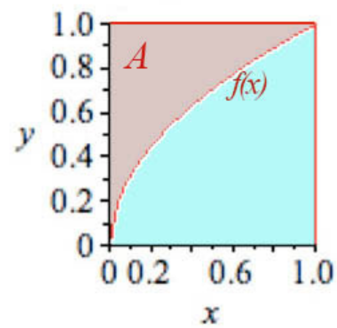
Then, the area, A, between the curves is

$$
\begin{aligned}
A = \int_{x_0}^{x_1} h(x)\, dx &= \int_{x_0}^{x_1} \left(\int_{y=f(x)}^{g(x)} dy \right) dx \\
&= \int_{x_0}^{x_1} \int_{f(x)}^{g(x)} dy\, dx \equiv \iint_A dA
\end{aligned}
\tag{9}
$$

where $dA = dx\, dy$ is a small chunk of *area*, integrated over the region, A, shown.

Original problem as a double integral

So, back to our original problem, which can be written in a new way. We evaluate the integrals from the inside out...



$$
\begin{aligned}
A = \iint_A dA &= \int_{x=0}^{1} \int_{y=\sqrt{x}}^{1} dy\, dx \\
&= \int_{x=0}^{x=1} \left(\int_{y=\sqrt{x}}^{1} 1\, dy \right) dx = \int_{x=0}^{x=1} \left(y\Big|_{y=\sqrt{x}}^{1} \right) dx \\
&= \int_{x=0}^{x=1} \left(1 - \sqrt{x}\right) dx = 1/3.
\end{aligned}
\tag{10}
$$

To do

Do Exercise 3 in `Lab 06`*, in which you'll convert the integral of Exercise 1 to a double integral over $dA = dy\, dx$.*

Then...

- ***Double Integrals and Areas***: *Write out each area as a double integral. You do not need to evaluate the integrals. For each area, you should be able to write the area as* ***one*** *double integral. (Hint: You may need to think about integrating* $x(y)\,dy$ *instead of* $y(x)\,dx$*.)*