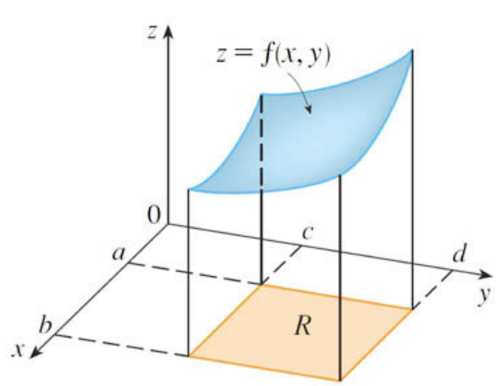
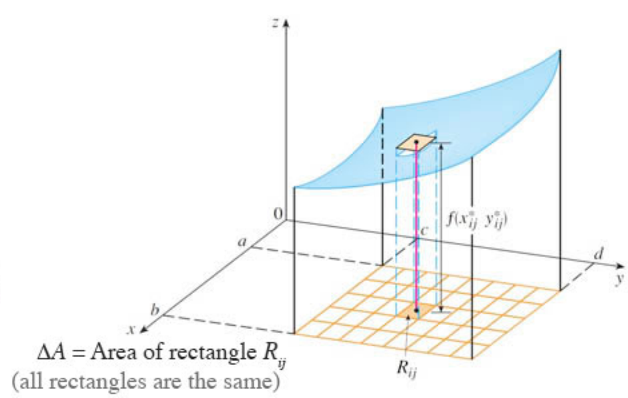
Double integrals over rectangles

Approximating volumes with stacks of blocks...

Finding volumes using double integrals

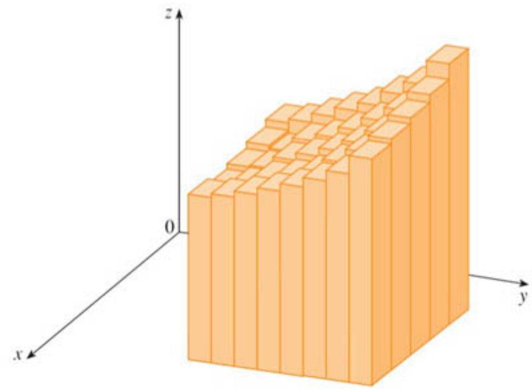
Consider the volume, V, of the solid which is...





- bounded below by the rectangle, $R = [a, b] \times [c, d]$ in the xy plane,

- Goes straight up from R,
- Is bounded above by the surface, $f(x,y)$.



- Split up $[a,b]$ into m sub-intervals, split up $[c,d]$ into n sub-intervals.

Approximate volume is

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A. \tag{1}$$

where... (x_{ij}^*, y_{ij}^*) is the coordinate of *some* point in rectangle R_{ij}; ΔA is the area of a rectangle with sides $\Delta x = (b-a)/m$ and $\Delta y = (d-c)/n$.

That is...

$$\Delta A = \frac{b-a}{m}\frac{d-c}{n} = \frac{A}{mn}, \tag{2}$$

where $A = (b-a)(d-c)$ is the area of the rectangular region R.

In the limit $\Delta A \to dA \to 0$ this sum becomes exactly this *integral*

$$V = \iint_R f(x,y)dA. \tag{3}$$

To do

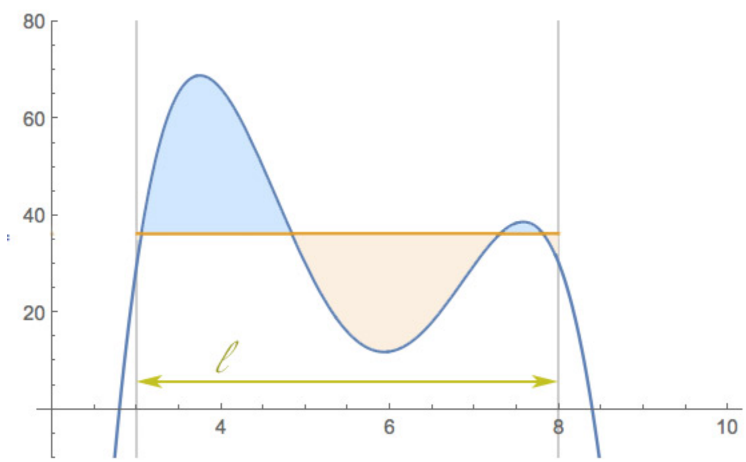
- *Double Integrals from Data*

$\langle f(x,y)\rangle$ above a rectangular region

Average value of a 1-dimensional function on the interval $[a, b]$:

$$\begin{aligned}\langle f(x)\rangle =& \frac{1}{b-a}\int_a^b f(x)\,dx \\ =& \frac{1}{l}\int_a^b f(x)\,dx\end{aligned} \tag{4}$$

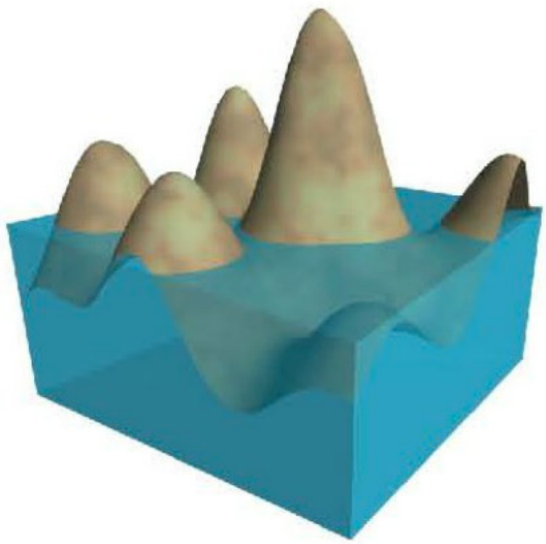
If you chopped off all the area in the peaks above this line at $y = \langle f(x)\rangle$, it would just fill in the area in the valleys below the line...



Analogously, for a function of x and y...

$$\begin{aligned}\langle f(x,y)\rangle =& \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\ =& \frac{1}{A}\int_a^b\int_c^d f(x,y)\,dy\,dx.\end{aligned} \tag{5}$$

If you chopped off all the volume in the peaks above this plane surface at $z = \langle f(x,y) \rangle$, it would just fill in the volume of the valleys below this plane.



Approximating the average by sampling

$$
\begin{aligned}
\langle f(x,y) \rangle =& \frac{1}{A} \int_a^b \int_c^d f(x,y)\, dy\, dx \\
=& \frac{1}{A} \sum_i^m \sum_j^n f(x_{ij}*, y_{ij}^*) \Delta A = \frac{\Delta A}{A} \sum_i^m \sum_j^n f(x_{ij}*, y_{ij}^*) \\
=& \frac{\Delta A * m * n}{A} \frac{\sum_{i=1}^m \sum_{j=1}^n f(x_{ij}*, y_{ij}^*)}{m * n} \\
=& \frac{\sum_i^m \sum_j^n f(x_{ij}*, y_{ij}^*)}{m * n},
\end{aligned}
$$

because there are $m * n$ sub-rectangles, so $A = \Delta A * m * n$. Therefore the average value $\langle f(x,y) \rangle$ is just the average of the sampled values of the function--one sample for each sub-rectangle.

To do

- *Back to the Park*