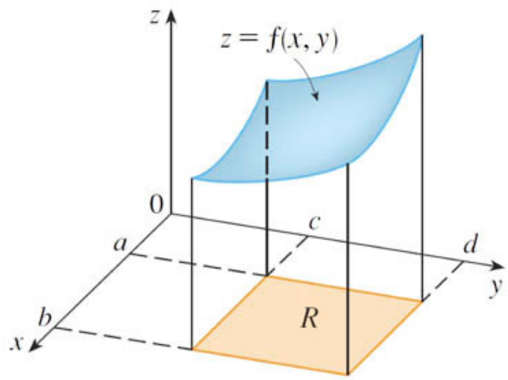
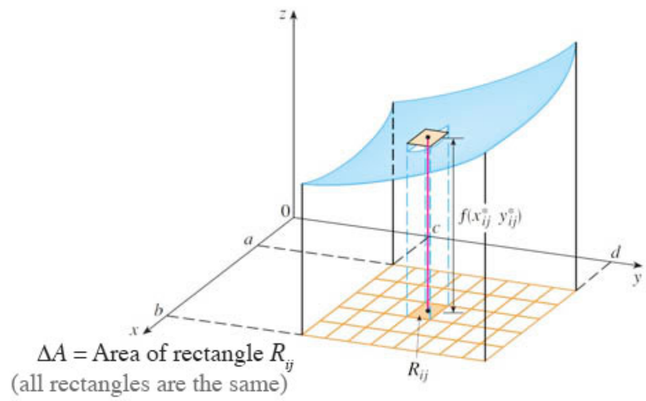
Iterated integrals & partial integration





We saw that the double integral:

$$\iint_R f(x,y)\,dA \tag{1}$$

represents the volume between the rectangle $R \equiv [a,b] \times [c,d]$ (in the x, y plane) and the surface $z = f(x,y)$ above R.

Now we'll see how to **evaluate** such an integral.

Partial Integration

How to evaluate a double integral like

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy? \tag{2}$$

We define **partial integration** of a function $f(x,y)$ with respect to y with this notation:

$$\int_{y=c}^{d} f(x,y)\,dy = A(x). \tag{3}$$

A(x) is the result of

- Treating x as a constant,
- carrying out the integration with respect to y.

example:

$$\begin{aligned}
\int_0^1 (2-x-y)\,dy =& \int_0^1 (2-x)\,dy - \int_0^1 y\,dy \\
=&(2-x)\int_0^1 dy - \int_0^1 y\,dy \\
=&(2-x)\,y\big|_0^1 - \frac{1}{2}y^2\bigg|_0^1 \\
=&(2-x)(1-0) - \frac{1}{2}(1^2-0^2) \\
A(x) =&2-x-\frac{1}{2} = \frac{3}{2}-x.
\end{aligned} \tag{4}$$

We write it as $A(x)$ because it is the **area** resulting from the partial integration w.r.t. y, but typically the result is *a function of* x. (See picture below for a visual interpretation of $A(x)$.)

Double integral

The double integral is the result, next, of carrying out the integration of $A(x)$ with respect to x:

$$\begin{aligned}\iint f(x,y)dA &= \int_{x=a}^{b} A(x)\,dx \\ &= \int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\,dy\right)\,dx \qquad (5) \\ &= \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx\end{aligned}$$

This rather sloppy way of writing the double integral leaves it unclear whether you should evaluate the partial integral of y first, and then integrate over x, or vice versa. Perhaps it doesn't matter?

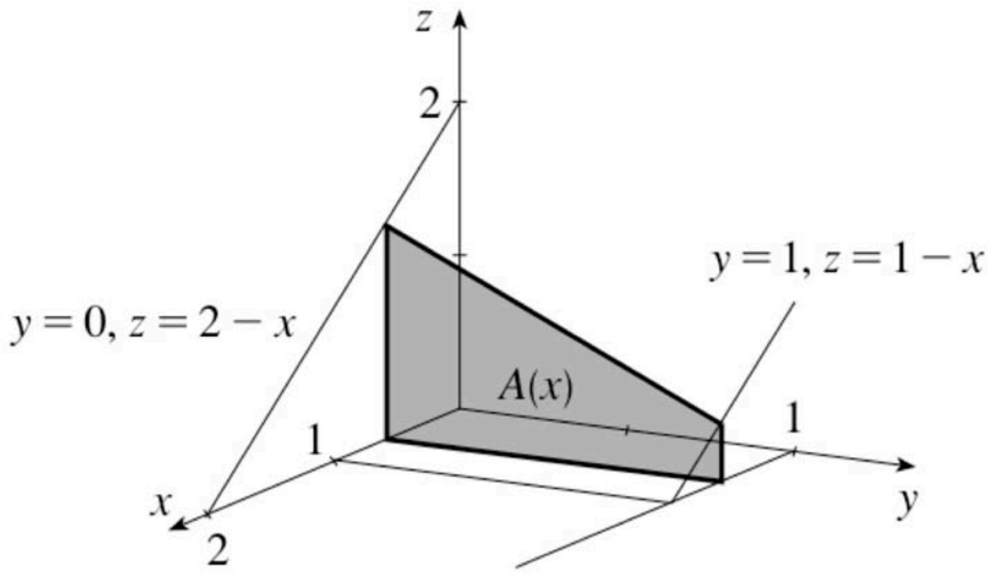
Fubini's Theorem

...says that for a rectangular region R and some pretty general assumptions, **the order of partial integration does not matter**.

Example 1

Find the volume underneath $f(x,y) = 2 - x - y$ and above the rectangle defined by points in the x-y-plane: $0 < x < 1$, and

$0 < y < 1$. (Alternately, $R = [0,1] \times [0,1]$.)

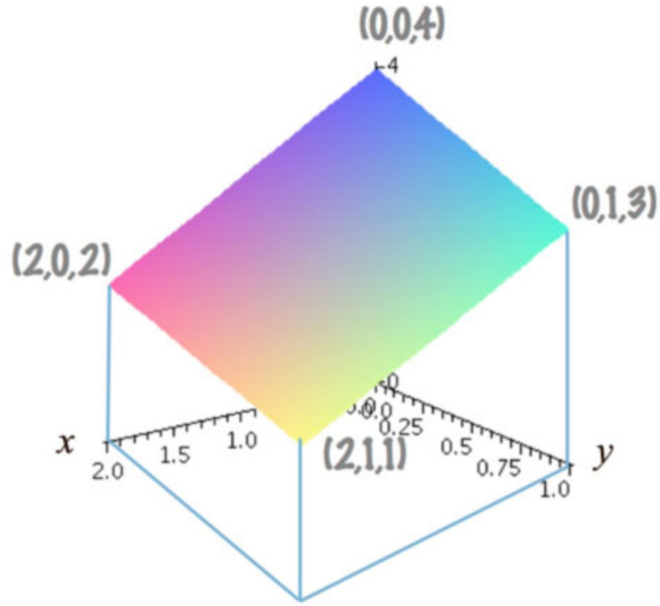


$$V = \iint (2 - x - y)\, dA = \int_{x=0}^{1} \int_{y=0}^{1} (2 - x - y) dy dx$$
$$= \int_{x=0}^{1} \left(\int_{y=0}^{1} (2 - x - y)\, dy \right) dx \tag{6}$$

We already carried out the partial integral in $(\ldots)$. Substituting the result back into the expression for V,

$$V = \int_{x=0}^{1} \left(\frac{3}{2} - x \right) dx = \frac{3}{2} \int_0^1 dx - \int_0^1 x\, dx$$
$$= \frac{3}{2} - \frac{1}{2} = 1 \tag{7}$$

Example 2



The function for the surface is $f(x, y) = 4 - x - y$.

- *Estimate the volume under this surface, for example, by estimating the average height of the surface, then multiplying by the area of the rectangle R underneath.*

- Evaluate the double integral to find the exact volume.

How to do this in CoCalc:

Carrying out:

$$\int_{x=0}^{2}\int_{y=0}^{1}(4-x-y)\,dy\,dx$$

```
var('x y')
integrate(   4-x-y,
          (y, 0, 1)
          )
```

```
-x + 7/2
```

```
integrate( _,
         (x,0,2))
```

```
5
```

How to do this in *Mathematica*.

To Do

- *Double integrals - practice: #1-4*

- *The shape of the solid*