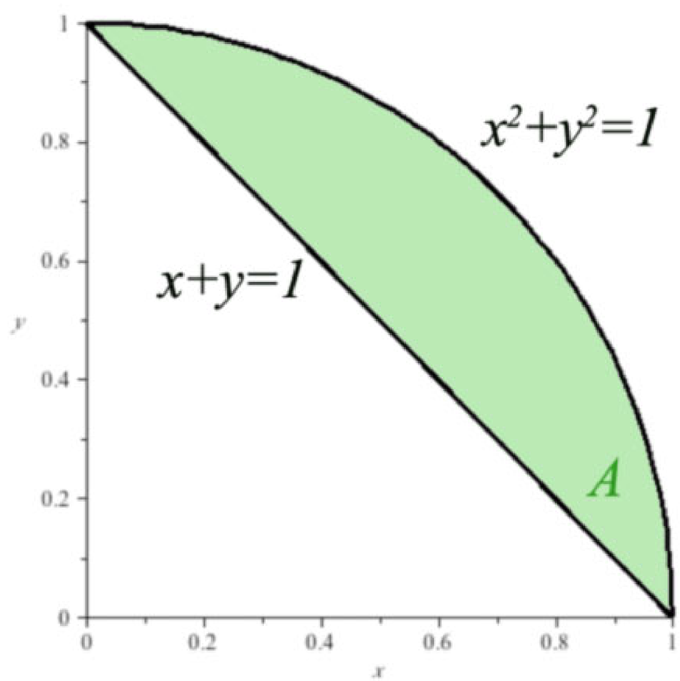
Double integrals over general areas

Connecting area and volume views

Or...connecting "area between curves" and "volume of a solid" interpretations of double-integrals.

Consider the green area in the xy plane



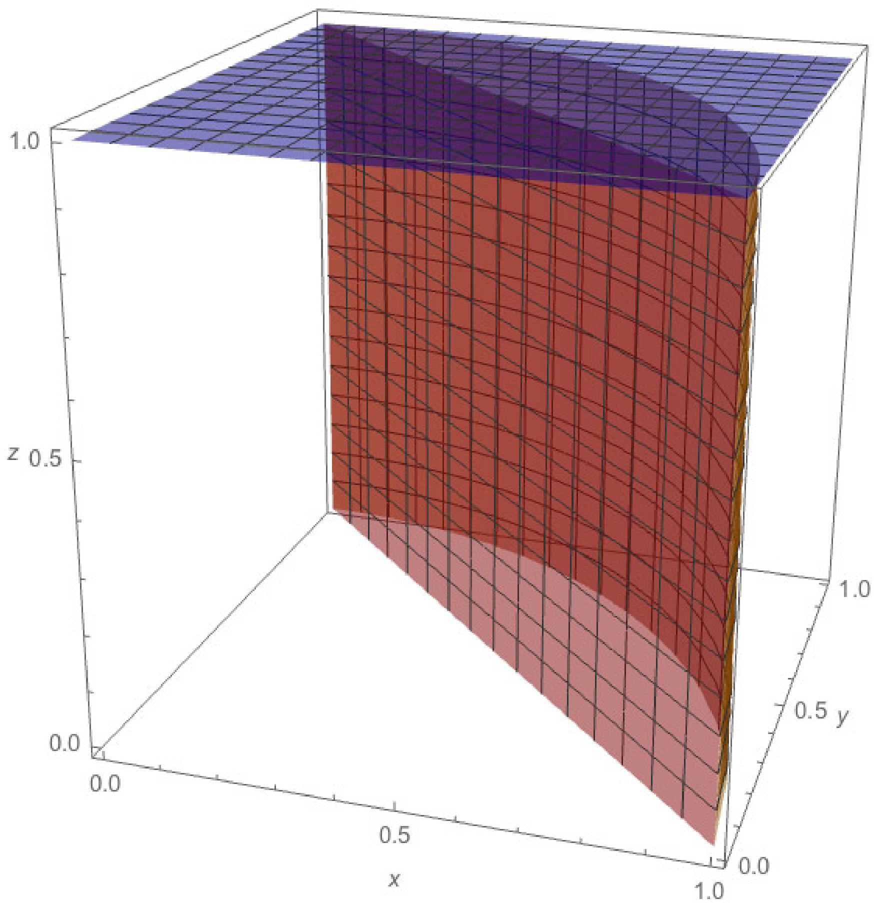
According to the "area between curves" view, the area in green is given by:

$$
\begin{aligned}
A =& \int_{x=0}^{1} \int_{y=1-x}^{\sqrt{1-x^2}} dA \\
=& \int_{x=0}^{1} \left(\int_{y=1-x}^{\sqrt{1-x^2}} dy \right) dx \\
=& \int_{x=0}^{1} \left(y\Big|_{y=1-x}^{\sqrt{1-x^2}} \right) dx \\
=& \int_{x=0}^{1} \left(\sqrt{1-x^2} - (1-x) \right) dx \\
=& \frac{\pi}{4} - \frac{1}{2}
\end{aligned}
\tag{1}
$$

But we could instead think about that double integral as the volume above a non-rectangular area in the xy plane, going up to the surface $f(x,y) = 1$, like this...

$$
\begin{aligned}
V =& \int_{x=0}^{1} \int_{y=1-x}^{\sqrt{1-x^2}} dA \\
=& \int_{x=0}^{1} \int_{y=1-x}^{\sqrt{1-x^2}} 1 \, dA
\end{aligned}
\tag{2}
$$

Then the volume corresponds to this solid:



- The base in the xy plane is the green area A,
- it extends straight up from A,
- to a height of 1...
- that is, it is bounded above by the surface $f(x, y) = 1$.

The value of the volume integral:

$$V = \int_{x=0}^{1} \int_{y=1-x}^{\sqrt{1-x^2}} 1 \, dy \, dz \tag{3}$$

is the same as the value of the area between curves that we first considered.

Double integrals above general areas

$$\iint_A f(x, y) \, dA = ? \tag{4}$$

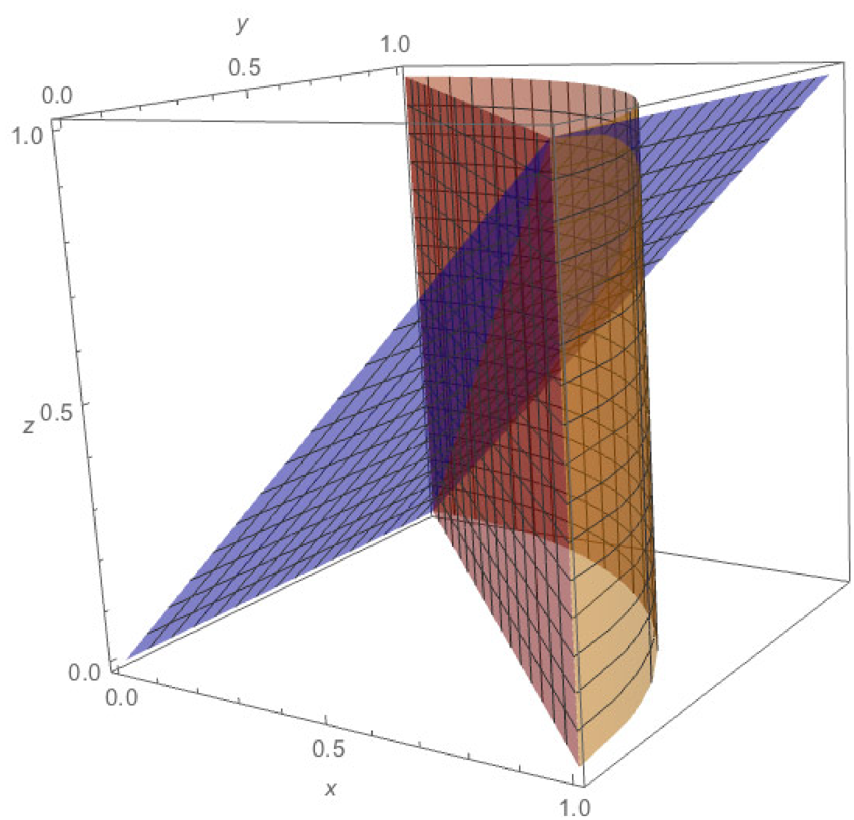
Let's generalize our interpretation of this integral as the volume under $f(x, y) = 1$ to *any* function $f(x, y)$: In general this double integral is...

- The volume of a solid,
- bounded, on the bottom, by a non-rectangular area, A, in the xy plane.
- going straight up from that area,

- bounded above by a surface $f(x, y)$

This volume integral is represented in the picture below:

$$V = \int_{x=0}^{1} \int_{y=1-x}^{\sqrt{1-x^2}} x \, dA \tag{5}$$



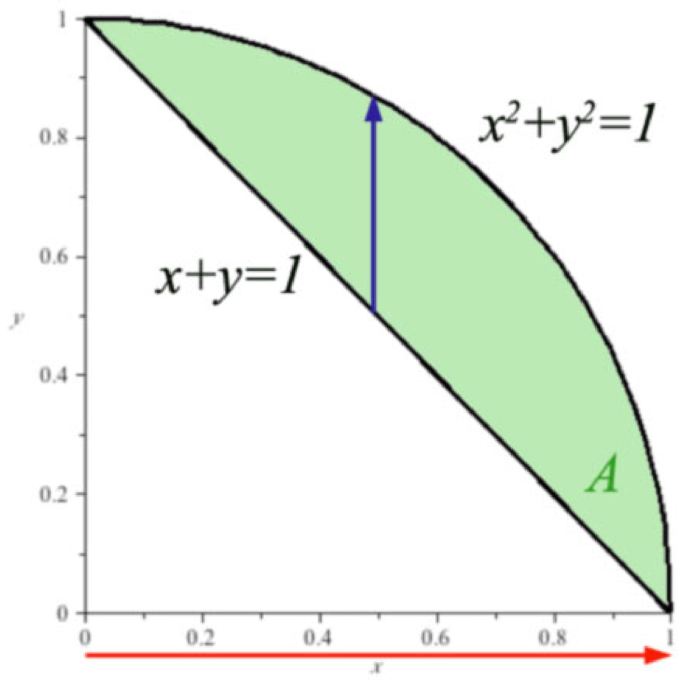
It's bounded by $f(x, y) = x$ above.

- See this [view of the surfaces](#) (GeoGebra).

We shall find that now, the order of integration *does* affect how we write down the limits of a double integral.

Order matters.

$\iint_A f(x,y)\,dA$



1. Integrate first with respect to y

2. This gives us the area, $A(x)$, of a slice through the solid above the blue arrow, *which depends on x,*

$$A(x) = \int_{y=1-x}^{\sqrt{1-x^2}} f(x,y)\,dy. \tag{6}$$

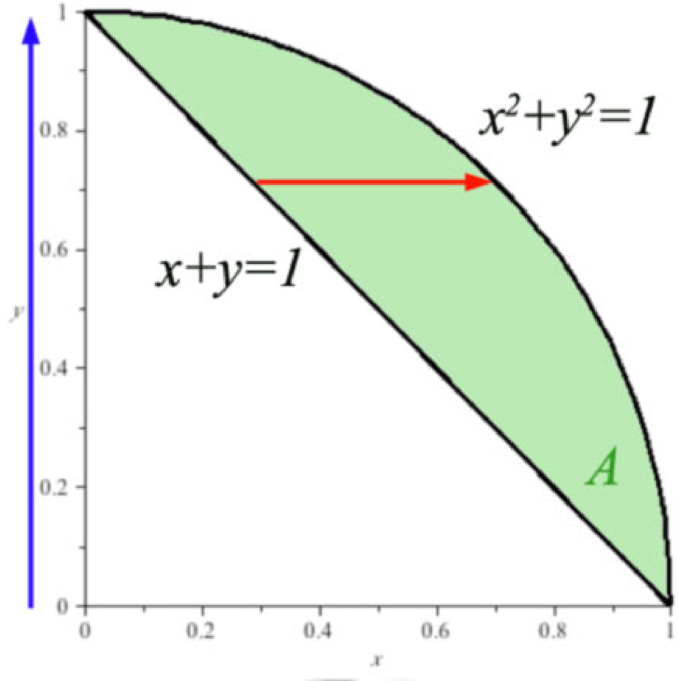
3. Integrate second with respect to x

Written...

$$\int_{x=0}^{1} A(x)\,dx = \int_{x=0}^{1}\int_{y=1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx \tag{7}$$

Order matters..

$\iint_A f(x,y)\,dA$



1. Integrate first with respect to x,

2. This gives us the area, $A(y)$, of a slice through the solid above the red arrow, *which depends on y,*

$$A(y) = \int_{x=1-y}^{\sqrt{1-y^2}} f(x,y)\,dx \tag{8}$$

3. Integrate second with respect to y

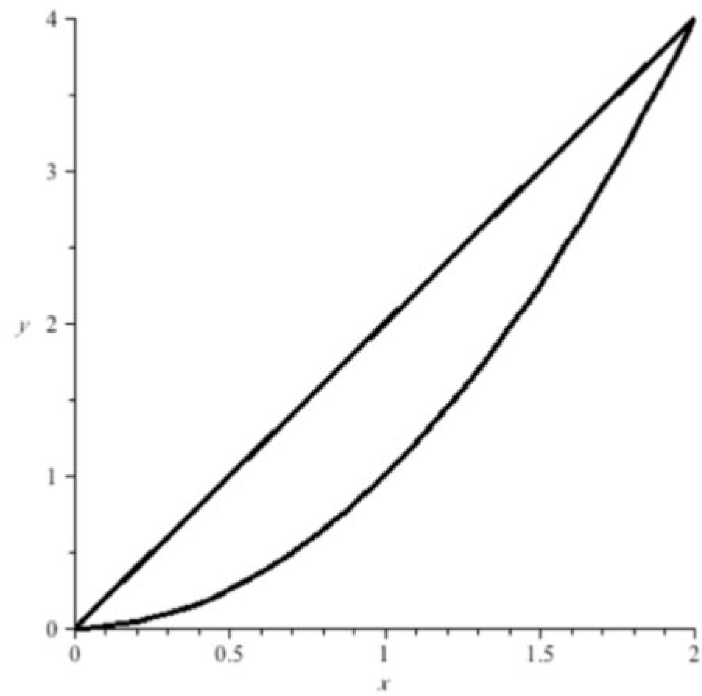
Written...

$$\int_{y=0}^{1} A(y)\,dy = \int_{y=0}^{1} \int_{x=1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy. \tag{9}$$

Example

$\iint_A (4x+2)\,dA$

Regions



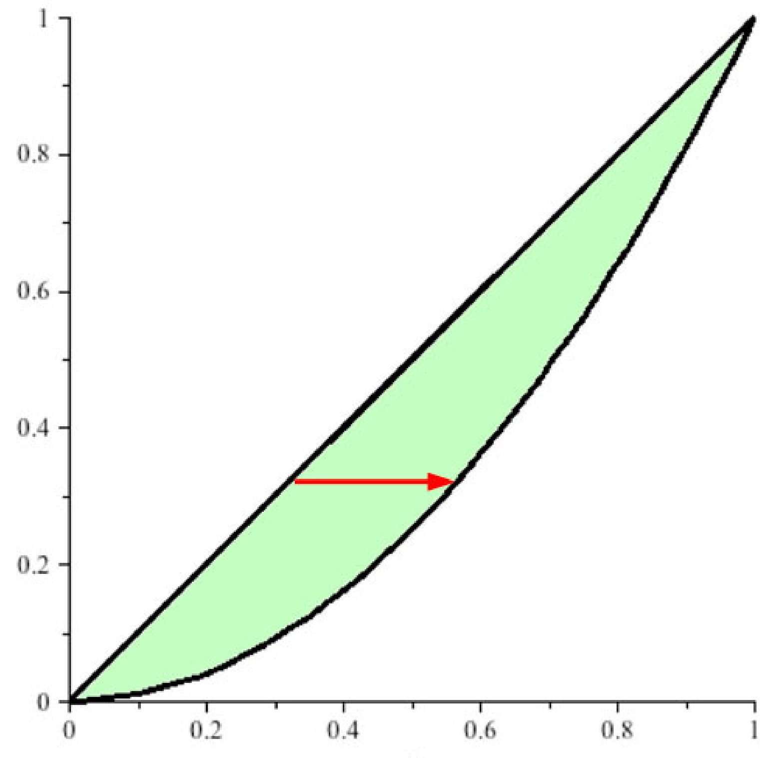
You should be able to sketch the region of integration in x and y given a double integral. For example:

$$\int_0^1 \int_y^{\sqrt{y}} x^2 y^2 \, dx \, dy \tag{10}$$

...means

$$\int_{y=0}^1 \left(\int_{x=y}^{\sqrt{y}} x^2 y^2 \, dx \right) \, dy \tag{11}$$

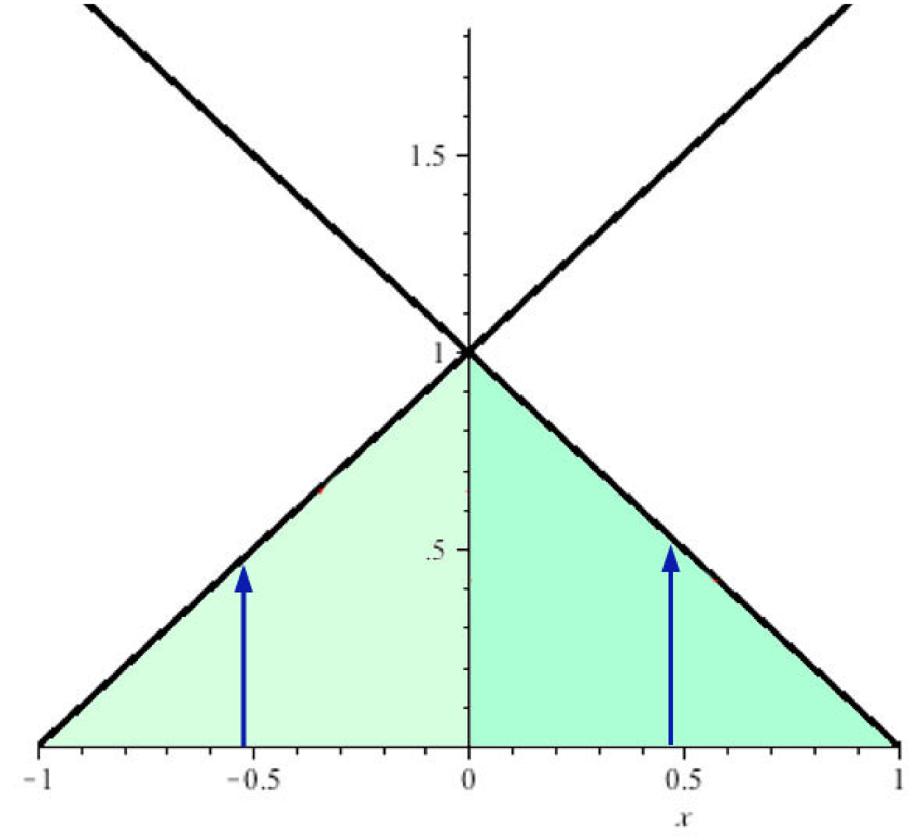
$\Rightarrow$



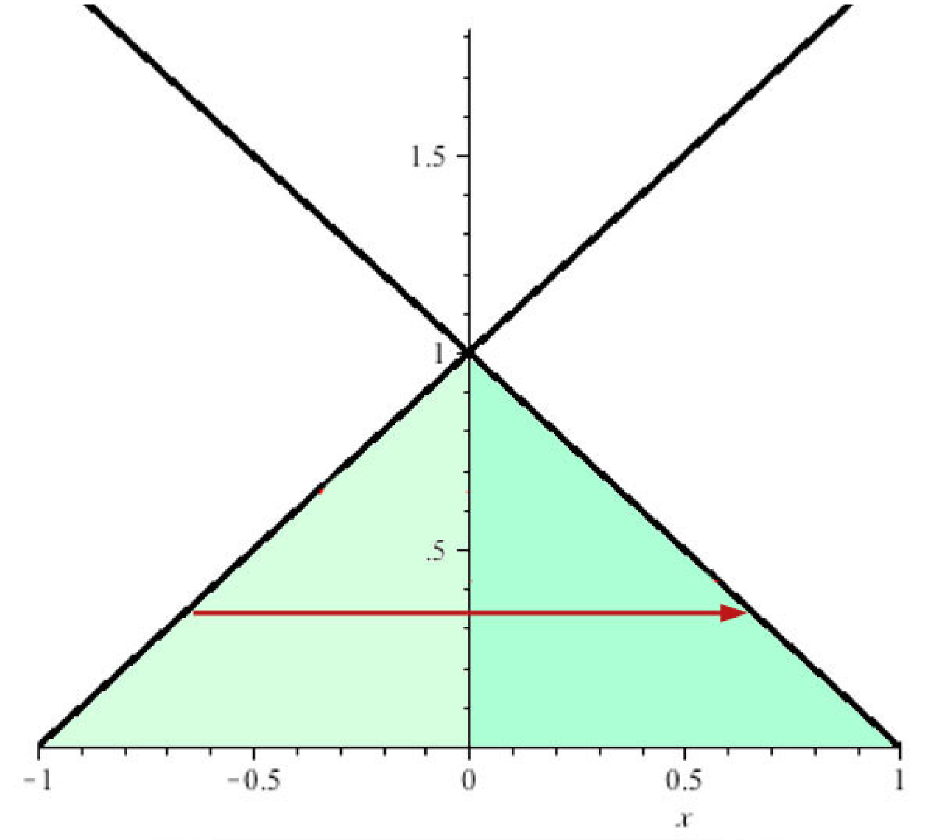
Changing the order of integration

$$\int_{x=-1}^{0}\int_{y=0}^{x+1} e^{x+y}\,dy\,dx + \int_{x=0}^{1}\int_{y=0}^{1-x} e^{x+y}\,dy\,dx \quad (12)$$

$\Rightarrow$



We could get the same result in one integral, integrating first over x then y like this...



The line $y = x + 1$ can be re-arranged to $x = y - 1$.

The line $y = 1 - x$ can be re-arranged to $x = 1 - y$:

$$\int_{y=0}^{1} \int_{x=y-1}^{1-y} e^{x+y} dx\, dy \tag{13}$$

To do

- Double Integrals Practice: #5-7

- Limits on Double Integrals

- Double Integrals: Problems