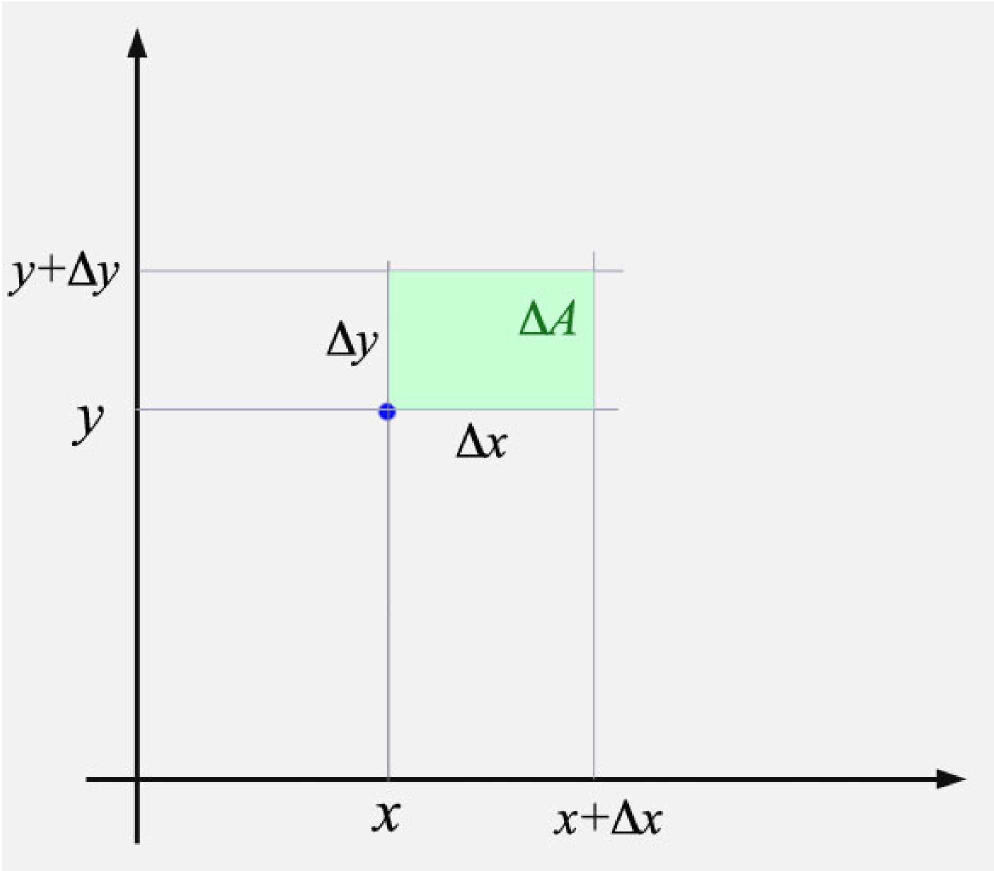
Polar integrals [12.4]

Kakslauttanen Igloo Village, Finland

Area in Cartesian coordinates

In Cartesian coordinates...

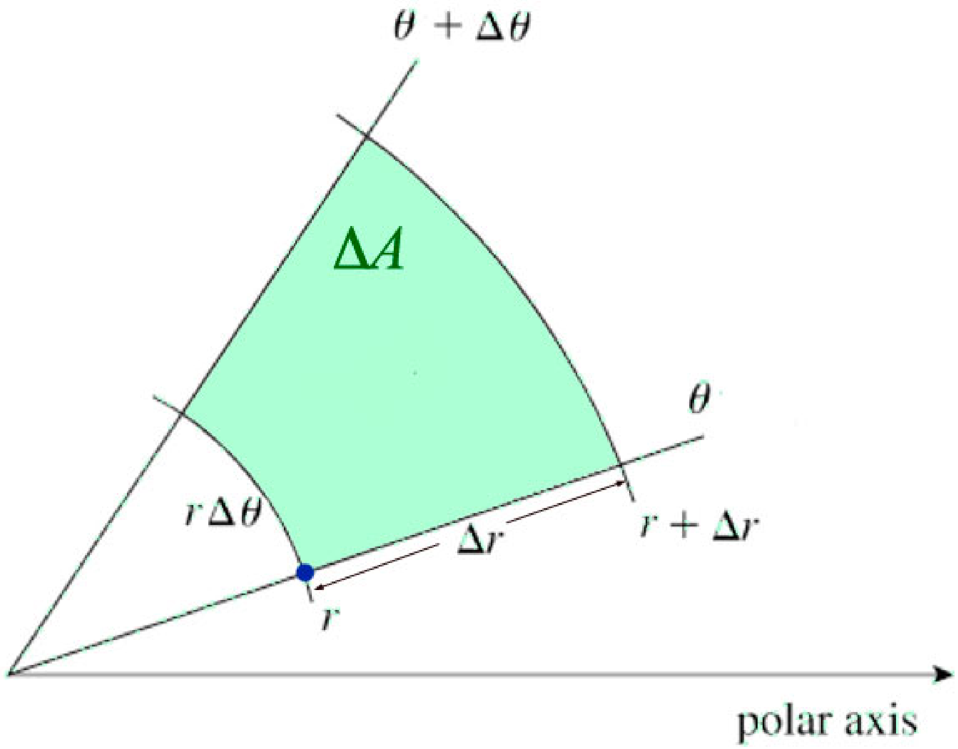


As $\Delta x \to dx$ and $\Delta y \to dy$, the rectangle has an area which approaches

$$dA = dx\, dy \tag{1}$$

Area in Polar coordinates

In polar coordinates...



As $\Delta\theta \to d\theta$ and $\Delta r \to dr$, the shaded region becomes more nearly rectangular, with sides of length $r\,d\theta$ and dr, so...

$$dA = r\,d\theta\,dr = r\,dr\,d\theta. \tag{2}$$

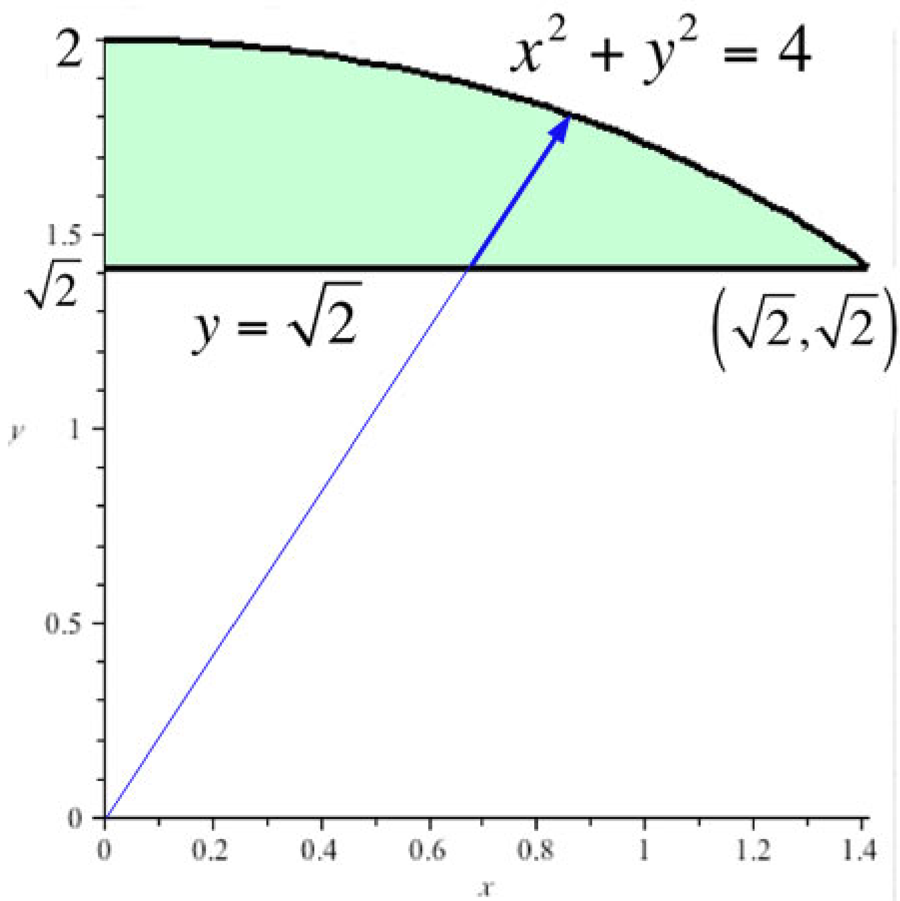
Area of a circle

The area of a circle of radius R can be found by this double integral:

$$\begin{aligned} A = \iint_A dA &= \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r\,dr\,d\theta \\ &= \int_{\theta=0}^{2\pi} \left(\frac{r^2}{2} \Big|_0^R \right) d\theta \\ &= \frac{R^2}{2} \int_{\theta=0}^{2\pi} d\theta = \frac{R^2}{2} 2\pi = \pi R^2. \end{aligned} \tag{3}$$

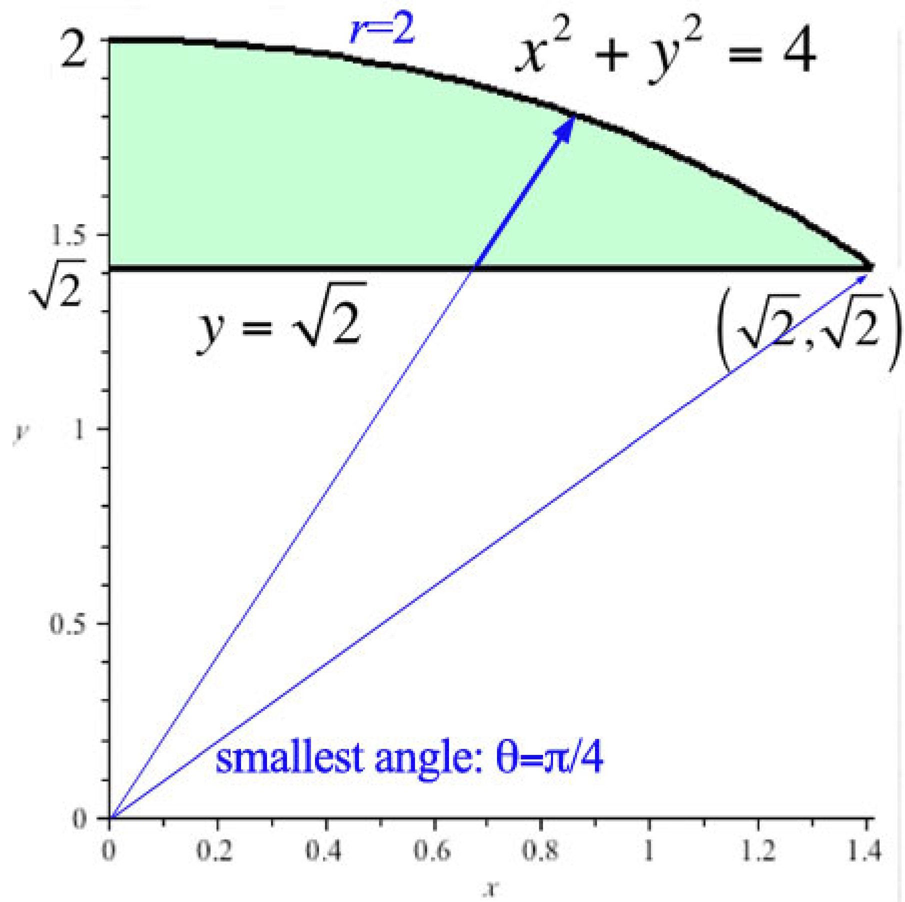
Double polar integrals.

$A = \iint_A f(r,\theta)\,dA = ?$



Double polar integrals..

$A = \iint_A f(r, \theta)\, dA = ?$



The line $\sqrt{2} = y = r\sin\theta$ becomes
$r(\theta) = \frac{\sqrt{2}}{\sin\theta} = \sqrt{2}\csc\theta.$

The double integral becomes:

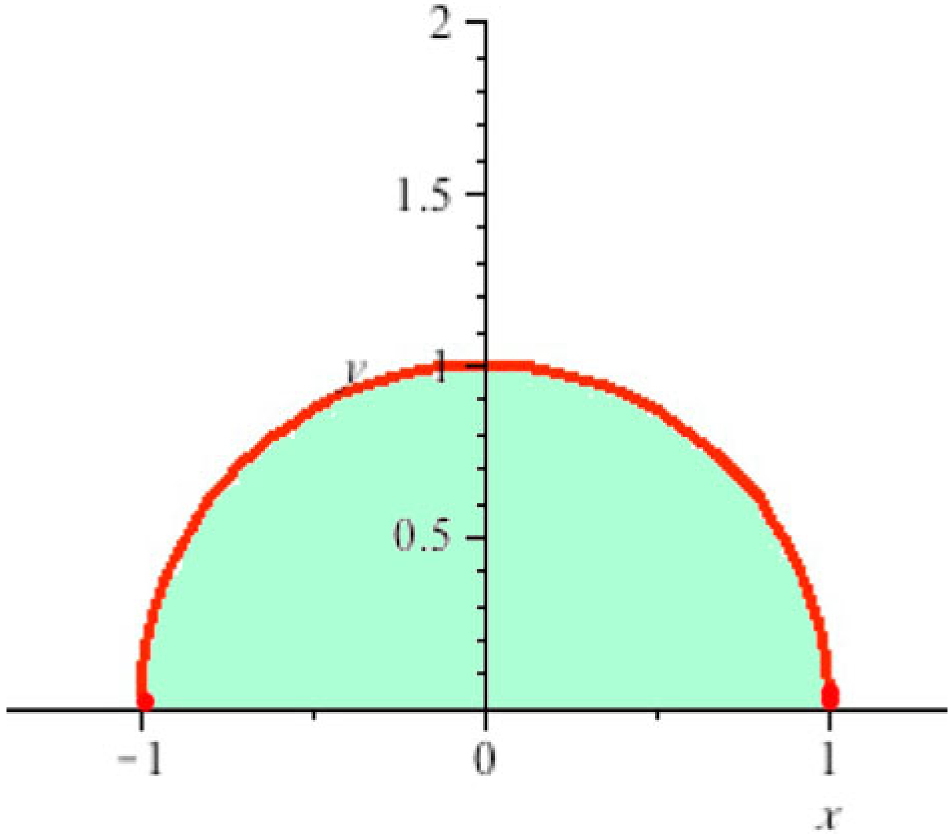
$$A = \int_{\theta=\pi/4}^{\pi/2} \int_{r=\sqrt{2}\csc\theta}^{2} f(r,\theta)\, r\, dr\, d\theta. \tag{4}$$

Converting...

Converting a double-integral in Cartesian coordinates to polar coordinates:

$\iint_A e^{x^2+y^2} dA = \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} e^{x^2+y^2} dy\, dx = \ldots$

- r runs from 0 to 1.
- θ runs from 0 to π.
- ah, $x^2 + y^2 = r^2$
- $dx\, dy = dA = r\, dr\, d\theta$



$$\begin{aligned}
\iint_A e^{r^2} dA &= \int_{\theta=0}^{\pi} \int_{r=0}^{1} e^{r^2} r\, dr\, d\theta \\
&= \int_{\theta=0}^{\pi} \left(\left. \frac{e^{r^2}}{2} \right|_{r=0}^{1} \right) d\theta \\
&= \left(\frac{e-1}{2} \right) \int_{\theta=0}^{\pi} d\theta = (e-1)\frac{\pi}{2} \approx 2.70
\end{aligned} \tag{5}$$

ToDo

Double integrals in polar coordinates

Image credits

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