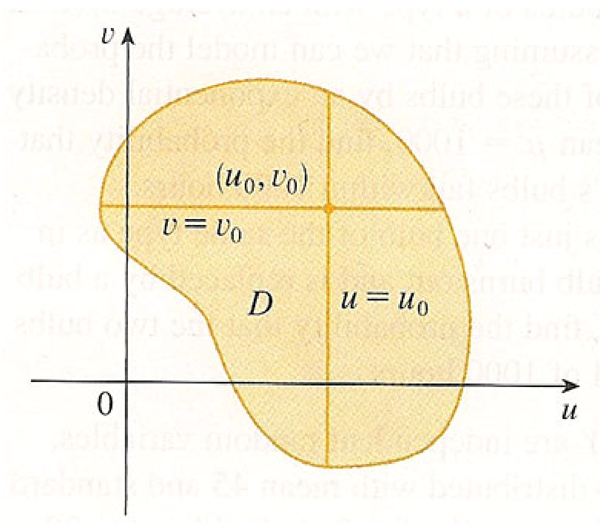
Parametric surfaces [12.6]

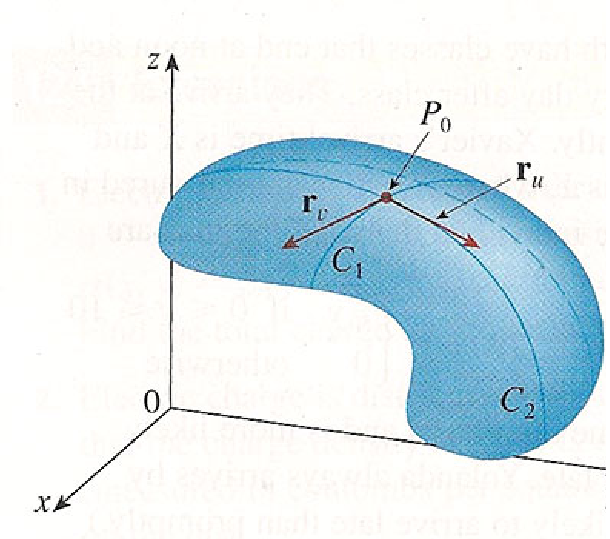
Review [Parametric Surfaces [10.5]]...

- As parameters u and v are varied in some domain D,
- The vector $\vec{\mathbf{r}}(u,v) = \langle\, x(u,v),\ y(u,y),\ z(u,v)\rangle$ traces out a surface S in 3D.

Tangent plane approximation

Outline of the tangent plane approximation:





- As v varies in D, (u is constant), $\vec{\mathbf{r}}$ traces the curve C_1. A tangent vector to C_1 is

$$\vec{\mathbf{r}}_v = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle \equiv \langle x_v, y_v, z_v\rangle. \tag{1}$$

  [What does that subscript mean in this context??]

- As u varies in D, (v is constant), $\vec{\mathbf{r}}$ traces the curve C_2. A tangent vector to C_2 is

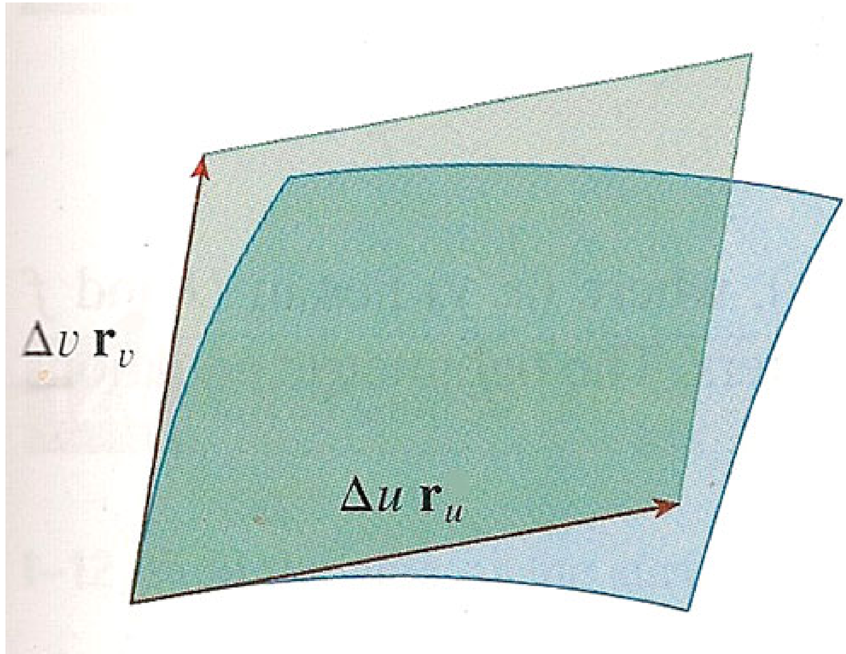
$$\vec{\mathbf{r}}_u = \langle x_u, y_u, z_u\rangle \tag{2}$$

- Increasing v to $v + \Delta v$ in D leads to a change of $\vec{\mathbf{r}}$ on the surface of *approximately* $\Delta v\, \vec{\mathbf{r}}_v$.
- Increasing u to $u + \Delta u$ in D leads to a change of $\vec{\mathbf{r}}$ on the surface of *approximately* $\Delta u\, \vec{\mathbf{r}}_u$.
- The area of the parallelogram shown is

  $|(\Delta v\, \vec{\mathbf{r}}_v) \times (\Delta u\, \vec{\mathbf{r}}_u)|$
  =$|\vec{\mathbf{r}}_v \times \vec{\mathbf{r}}_u|\Delta A$

  Where $\Delta u \Delta v = \Delta A$, an area in D.

- This cross product is *approximately* equal to ΔS, the area of a "patch" of the surface.



In the limit $\Delta u \to du,\ \Delta v \to du$, the approximation gets more exact, and the surface area of S is

$$S = \iint_D dS = \iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA. \tag{3}$$

[Later on, we'll consider surface integrals [13.6]. It will be useful that the direction of the cross-product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is perpendicular to (normal to) the surface.]

Surface area of a graph defined in terms of (x, y)

The general expression for surface area:

$$S = \iint_D dS = \iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA. \tag{4}$$

For the special case of the surface $z = f(x, y)$, $u \to x$, $v \to y$, such that (x, y) is in the domain D,

$$\begin{aligned} \vec{\mathbf{r}}_u \to \vec{\mathbf{r}}_x =& \langle \frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x} \rangle \\ =& \langle 1, 0, \frac{\partial z}{\partial x} \rangle \end{aligned} \tag{5}$$

Since x and y are at right angles (orthogonal), a change in x does not change y, and vice versa. So $\frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0$.

And,

$$\vec{\mathbf{r}}_v \to \vec{\mathbf{r}}_y = \langle 0, 1, \frac{\partial z}{\partial y} \rangle \tag{6}$$

Writing out the cross product of the two vectors:

$$\begin{aligned} \vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y =& \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{pmatrix} \\ =& -\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}. \end{aligned} \tag{7}$$

And therefore the surface area can be written as:

$$A_S = \iint_D |\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y| \, dA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA \tag{8}$$

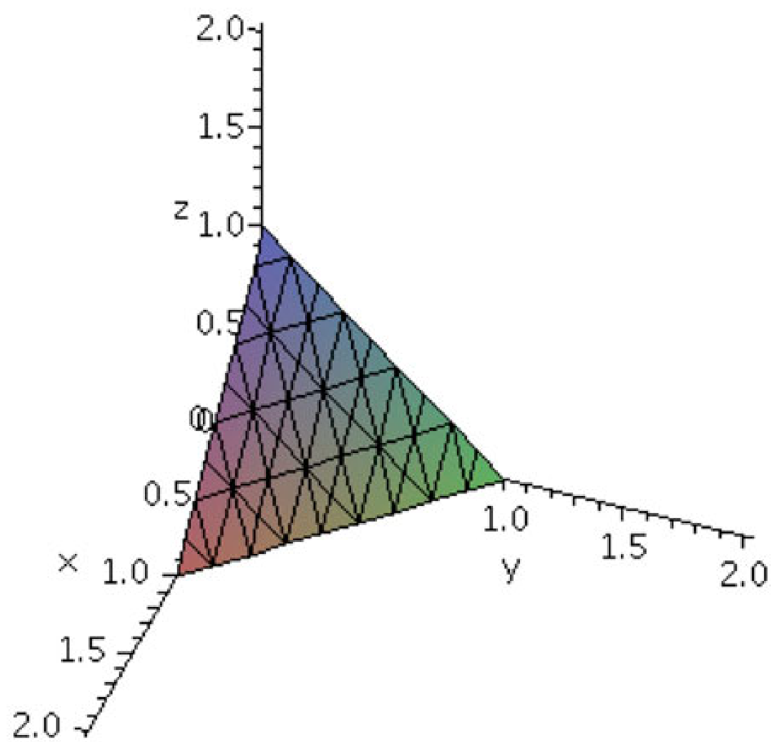
Surface area of a graph defined in terms of (x, y)

For the special case of the surface $z = f(x, y)$, $u \to x$, $v \to y$, such that (x, y) is in the domain D, the surface area can be written (see *below) :

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA \tag{9}$$

Example

Area of the surface shown of the plane: $z = 1 - x - y$



$$\begin{aligned} S &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA \\ &= \iint_D \sqrt{1 + (-1)^2 + (-1)^2} \, dA = \iint_D \sqrt{3} \, dA \\ &= \sqrt{3} \iint_D dA = \sqrt{3} A_D = \sqrt{3} * (1 * 1)/2 = \sqrt{3}/2 \end{aligned} \tag{10}$$

Check the result by using geometry to calculate the surface area...