Characterizing vector fields

Some vector fields, $\vec{\mathbf{F}}$, represent the gradient of a function, $\vec{\mathbf{F}} = \vec{\nabla} f$.

In the next few class sessions we'll be trying to figure out:

- How to tell if a vector field is the gradient of a function,
- How to find the function f, given $\vec{\mathbf{F}}$.
- How this makes some line integrals *a lot* easier to evaluate.

Gradients

The gradient of a scalar field is a vector field.

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} = \vec{\mathbf{v}}(x,y) \tag{1}$$

$$\vec{\nabla} f(x,y,z) = f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}} = \vec{\mathbf{v}}(x,y,z) \tag{2}$$

Gradient Fields

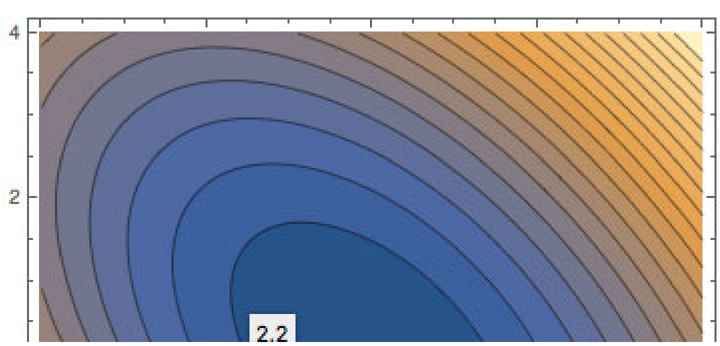
Some (but not all) vector fields are **gradient fields**. Consider $f(x,y) = x^2 + xy + y^2$. The gradient is

$$\vec{\nabla} f = \langle f_x, f_y \rangle = (2x+y)\,\hat{\mathbf{i}} + (2y+x)\,\hat{\mathbf{j}}. \tag{3}$$

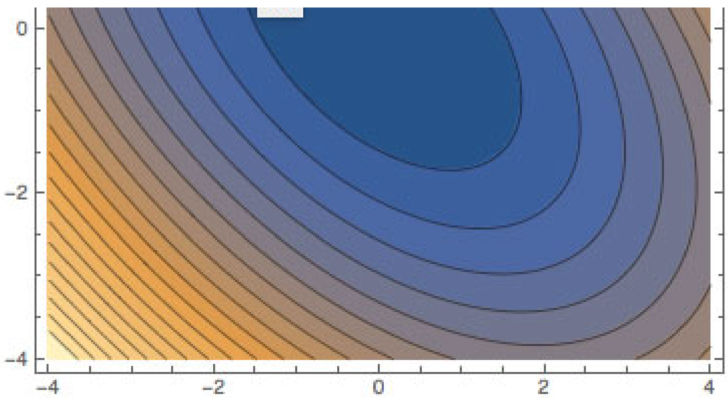
Graphing the function f below in contour plot. What can you say about

- *The direction of the vectors in the vector field $\vec{\mathbf{v}} = \vec{\nabla} f$?*
- *Where are the vectors representing $\vec{\mathbf{v}}$ longer or shorter?*

```
In[23]:= mycp = ContourPlot[x^2 + x y + y^2, {x, -4, 4},
    {y, -4, 4}, Contours → 20]
```

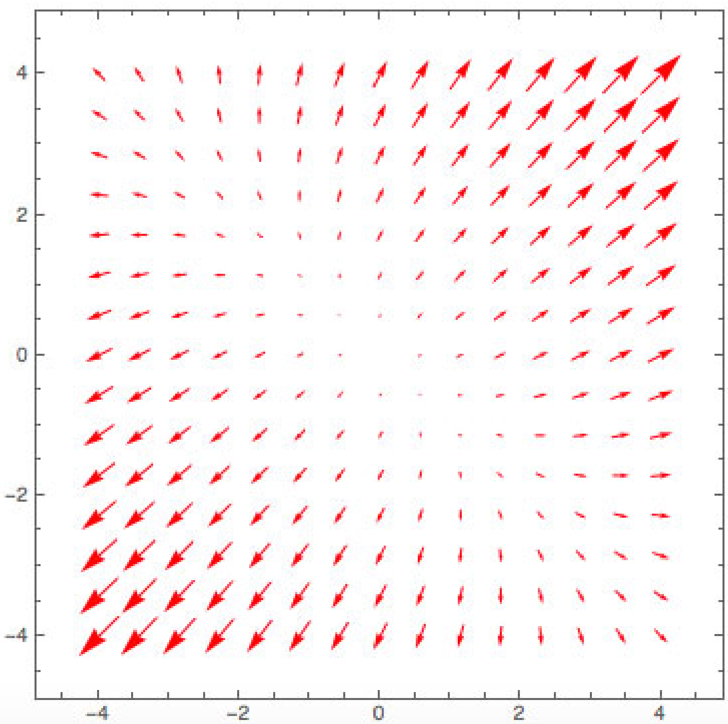


Out[23]=



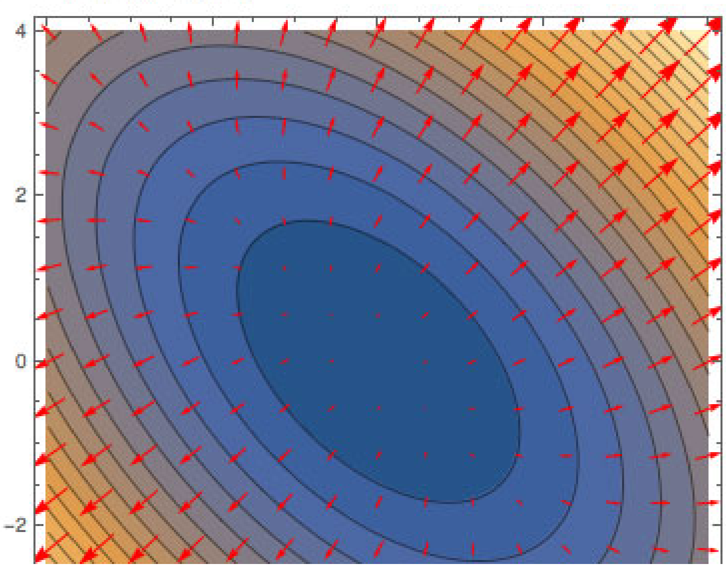
In[30]:=
```
myvp = VectorPlot[  {2 x + y, 2 y + x}   , {x, -4, 4},
  {y, -4, 4}, VectorStyle -> Red]
```

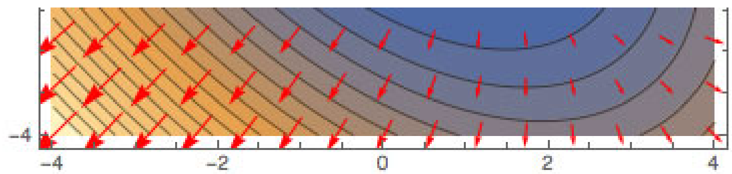
Out[30]=



In[31]:=
```
Show[mycp, myvp]
```

Out[31]=





In CoCalc

You'll see how to plot vector fields on top of contour plots in **Lab08**.

Conservative fields

Consider the vector field...

$$\vec{\mathbf{F}}(x,y) = 2x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \tag{4}$$

Can you guess a function $f(x,y)$ such that

$$\vec{\mathbf{F}}(x,y) = \vec{\nabla} f(x,y)? \tag{5}$$

In other words, what function, $f(x,y)$ has partial derivatives $f_x = 2x$ and $f_y = y$?

If such a function f exists:

- $f(x,y)$ is called a **potential function** for the vector field $\vec{\mathbf{F}}$.
- $\vec{\mathbf{F}}(x,y)$ is a **conservative vector field**.