be a ***vector*** denoted $\mathbf{u} \times \mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is determined by the right-hand rule: if we point the index finger of our right hand in the direction of $\mathbf{u}$ and our middle finger in the direction of $\mathbf{v}$, then our thumb points in the direction of $\mathbf{u} \times \mathbf{v}$.

a. We begin by defining the cross products using the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Referring to Figure 9.4.1, explain why $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in that order form a right-hand system. We then define $\mathbf{i} \times \mathbf{j}$ to be $\mathbf{k}$ — that is $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

b. Now explain why $\mathbf{i}$, $\mathbf{k}$, and $-\mathbf{j}$ in that order form a right-hand system. We then define $\mathbf{i} \times \mathbf{k}$ to be $-\mathbf{j}$ — that is $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

c. Continuing in this way, complete the missing entries in Table 9.4.2.

Table 9.4.2. *Table of cross products involving* $\mathbf{i}$, $\mathbf{j}$, *and* $\mathbf{k}$.

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$	$\mathbf{j} \times \mathbf{k} = +\hat{i}$
$\mathbf{j} \times \mathbf{i} = -\hat{k}$	$\mathbf{k} \times \mathbf{i} = +\hat{j}$	$\mathbf{k} \times \mathbf{j} = -\hat{i}$

d. Up to this point, the products you have seen, such as the product of real numbers and the dot product of vectors, have been commutative, meaning that the product does not depend on the order of the terms. For instance, $2 \cdot 5 = 5 \cdot 2$. The table above suggests, however, that the cross product is *anti-commutative*: for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. If we consider the case when $\mathbf{u} = \mathbf{v}$, this shows that $\mathbf{v} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{v})$. What does this tell us about $\mathbf{v} \times \mathbf{v}$; in particular, what vector is unchanged by scalar multiplication by -1? Only the 0 vector (zero)

e. It is not difficult to show that the cross product interacts with scalar multiplication and vector addition as one would expect: that is

$$(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v})$$
$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$$

We can combine these properties to make cross product calculations a bit easier. For example,

$$\begin{aligned}(2\mathbf{i} + \mathbf{j}) \times \mathbf{k} &= (2\mathbf{i} \times \mathbf{k}) + (\mathbf{j} \times \mathbf{k}) \\ &= 2(\mathbf{i} \times \mathbf{k}) + (\mathbf{j} \times \mathbf{k}) \\ &= -2\mathbf{j} + \mathbf{i}.\end{aligned}$$

Using these properties along with Table 9.4.2, find the cross product $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{k}$.

f. Verify that the cross product $\mathbf{u} \times \mathbf{v}$ you just found in part (e) is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

g. Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ in the xy-plane as shown below in e.) $(2\hat{\imath} + 3\hat{\jmath}) \times (-\hat{\imath} + \hat{k})$

$= -2\,\underbrace{\hat{\imath}\times\hat{\imath}} + 2\,\underbrace{\hat{\imath}\times\hat{k}} - 3\hat{\jmath}\times\hat{\imath} + 3\hat{\jmath}\times\hat{k}$

$= -2\cdot 0 + 2(-\hat{\jmath}) - 3(-\hat{k}) + 3\hat{\imath}$

$= 3\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

f.) $\vec{u}\cdot(\vec{u}\times\vec{v}) = \langle 2,3,0\rangle \cdot \langle 3,-2,3\rangle$

$= 2\cdot 3 - 3\cdot 2 + 0 = 0$ ✓

$\vec{v}\cdot(\vec{u}\times\vec{v}) = \langle -1,0,1\rangle \cdot \langle 3,-2,3\rangle$

$= -1\cdot 3 + 0\cdot -2 + 1\cdot 3 = 0$ ✓

g.) $\vec{u}\times\vec{v} = u\cdot\hat{\imath} \times (v\cos\theta\,\hat{\imath} + v\sin\theta\,\hat{\jmath})$

$= uv\cos\theta\,\underbrace{\hat{\imath}\times\hat{\imath}}_{0} + uv\sin\theta\,\underbrace{\hat{\imath}\times\hat{\jmath}}_{\hat{k}}$

$= uv\sin\theta\,\hat{k}$

So $|\vec{u}\times\vec{v}| = uv\sin\theta$

Figure 9.4.3.

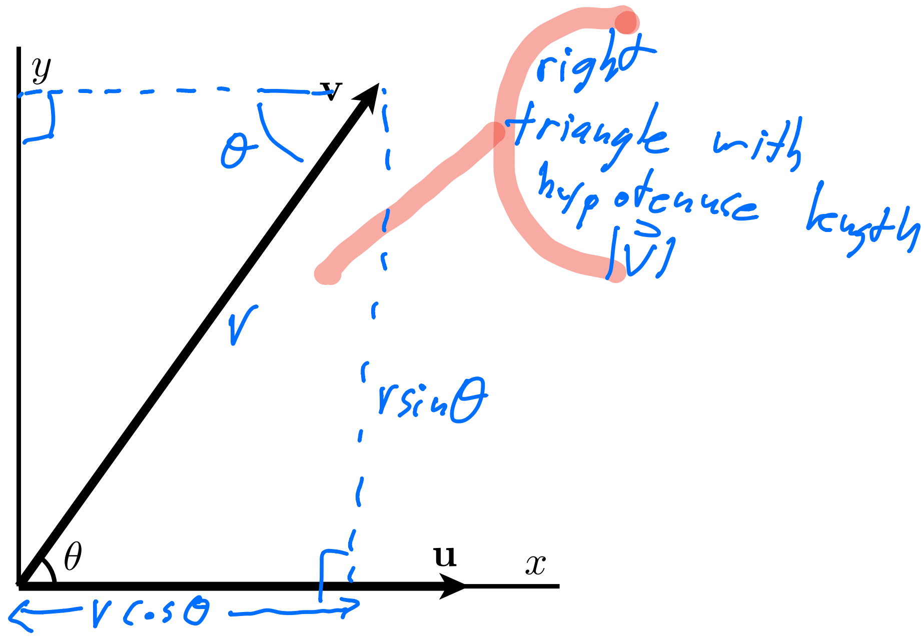

Figure 9.4.3. Two vectors in the xy-plane

Explain why $\mathbf{u} = |\mathbf{u}|\mathbf{i}$ and $\mathbf{v} = |\mathbf{v}|\cos(\theta)\mathbf{i} + |\mathbf{v}|\sin(\theta)\mathbf{j}$. Then compute the length of $|\mathbf{u} \times \mathbf{v}|$.

9.4.1 Computing the cross product

As we have seen in Preview Activity 9.4.1, the cross product $\mathbf{u} \times \mathbf{v}$ is defined for two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ and produces another vector in $\mathbb{R}^3$. Using the right-hand rule, we saw that

$$\begin{aligned} &\mathbf{i} \times \mathbf{j} = \mathbf{k} && \mathbf{i} \times \mathbf{k} = -\mathbf{j} && \mathbf{j} \times \mathbf{k} = \mathbf{i} \\ &\mathbf{j} \times \mathbf{i} = -\mathbf{k} && \mathbf{k} \times \mathbf{i} = \mathbf{j} && \mathbf{k} \times \mathbf{j} = -\mathbf{i} \end{aligned}$$

If, in addition, we assume the cross product behaves like we think a product should (e.g., the cross product distributes over vector addition), we can compute the cross product in terms of the components of general vectors to find a formula for the cross product. Doing so we see that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1\mathbf{i} \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) + u_2\mathbf{j} \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &\quad + u_3\mathbf{k} \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}. \end{aligned}$$

(Like the dot product, the cross product arises in physical applications, e.g.,