9.5 Lines and Planes in Space

Motivating Questions

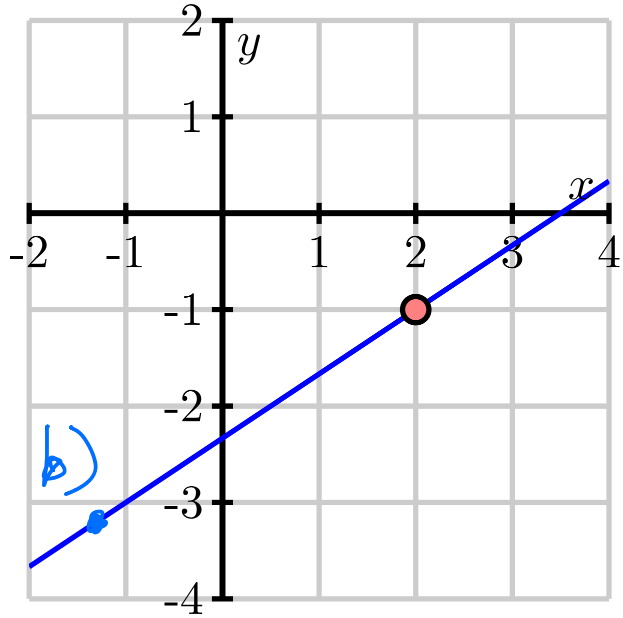
- How are lines in $\mathbb{R}^3$ similar to and different from lines in $\mathbb{R}^2$?
- What is the role that vectors play in representing equations of lines, particularly in $\mathbb{R}^3$?
- How can we think of a plane as a set of points determined by a point and a vector?
- How do we find the equation of a plane through three given non-collinear points?

In single variable calculus, we learn that a differentiable function is *locally linear*. In other words, if we zoom in on the graph of a differentiable function at a point, the graph looks like the tangent line to the function at that point. Linear functions played important roles in single variable calculus, useful in approximating differentiable functions, in approximating roots of functions (see Newton's Method), and approximating solutions to first order differential equations (see Euler's Method). In multivariable calculus, we will soon study curves in space; differentiable curves turn out to be locally linear as well. In addition, as we study functions of two variables, we will see that such a function is locally linear at a point if the surface defined by the function looks like a plane (the tangent plane) as we zoom in on the graph.

Consequently, it is important for us to understand both lines and planes in space, as these define the linear functions in $\mathbb{R}^2$ and $\mathbb{R}^3$. (Recall that a function is linear if it is a polynomial function whose terms all have degree less than or equal to 1. For example, x defines a single variable linear function and $x + y$ a two variable linear function, but xy is not linear since it has degree two, the sum of the degress of its factors.) We begin our work by considering some familiar ideas in $\mathbb{R}^2$ but from a new perspective.

Preview Activity 9.5.1. We are familiar with equations of lines in the plane in the form $y = mx + b$, where m is the slope of the line and $(0, b)$ is the y-intercept. In this activity, we explore a more flexible way of representing lines that we can use not only in the plane, but in higher dimensions as well.

To begin, consider the line through the point $(2, -1)$ with slope $\frac{2}{3}$ as shown in Figure 9.5.1.



slope, $m = \frac{\Delta y}{\Delta x}$

$= \frac{1}{1.5} = \frac{2}{3}$

$\frac{\Delta y}{\Delta x}$

$\Delta y = m \cdot \Delta x$

Figure 9.5.1. The line through $(2, -1)$ with slope $\frac{2}{3}$.

$\Delta y = \frac{2}{3}$

a. Suppose we increase x by 1 from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate 3? $(3, -\frac{1}{3})$

b. Suppose we decrease x by 3.25 from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate -1.25?

c. Now, suppose we increase x by some arbitrary value $3t$ from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate $2 + 3t$?

d. Observe that the slope of the line is related to any vector whose y-component divided by the x-component is the slope of the line. For the line in this exercise, we might use the vector $\langle 3, 2 \rangle$, which describes the direction of the line. Explain why the terminal points of the vectors $\mathbf{r}(t)$, where

$$\mathbf{r}(t) = \langle 2, -1 \rangle + \langle 3, 2 \rangle t,$$

trace out the graph of the line through the point $(2, -1)$ with slope $\frac{2}{3}$.

e. Now we extend this vector approach to $\mathbb{R}^3$ and consider a second example. Let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through the point $(1, 0, 2)$ in the direction of the vector $\langle 2, -1, 4 \rangle$. Find the coordinates of three distinct points on line $\mathcal{L}$. Explain your thinking.

b.) $\Delta y = \underset{\Delta x}{-3.25} \cdot m = -\frac{2}{3} \cdot 3.25 = \frac{-6.5}{3} = -2\frac{1}{6} \approx -2.17$

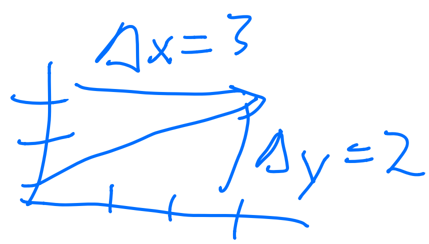
$\left(-1.25, \begin{pmatrix} -2.17 \\ -1 \end{pmatrix}\right)$

$= (-1.25, -3.17)$ (see above)

c.) $\Delta y = \underset{\Delta x}{3t} \cdot m = 3t \cdot \frac{2}{3} = 2t$

$(2+3t, -1+2t)$

d.) vector $\langle 3, 2 \rangle$ =



slope of line is $\frac{\Delta y}{\Delta x} = \frac{2}{3}$

e.) $\vec{r} = (1, 0, 2) + t\langle 2, -1, 4 \rangle$

f.) (try different values of t)

f. Find a vector in the form

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t$$

whose terminal points trace out the line $\mathcal{L}$ that is described in (e). That is, you should be able to locate any point on the line by determining a corresponding value of t.

9.5.1 Lines in Space

In two-dimensional space, a non-vertical line is defined to be the set of points satisfying the equation

$$y = mx + b,$$

for some constants m and b. The value of m (the slope) tells us how the dependent variable changes for every one unit increase in the independent variable, while the point $(0, b)$ is the y-intercept and anchors the line to a location on the y-axis. Alternatively, we can think of the slope as being related to the vector $\langle 1, m \rangle$, which tells us the direction of the line, as shown on the left in Figure 9.5.2. Thus, we can identify a line in space by fixing a point P and a direction $\mathbf{v}$, as shown on the right. Since we also have vectors in space to provide direction, this same idea of a point and a direction determining a line works in $\mathbb{R}^n$ for any n.

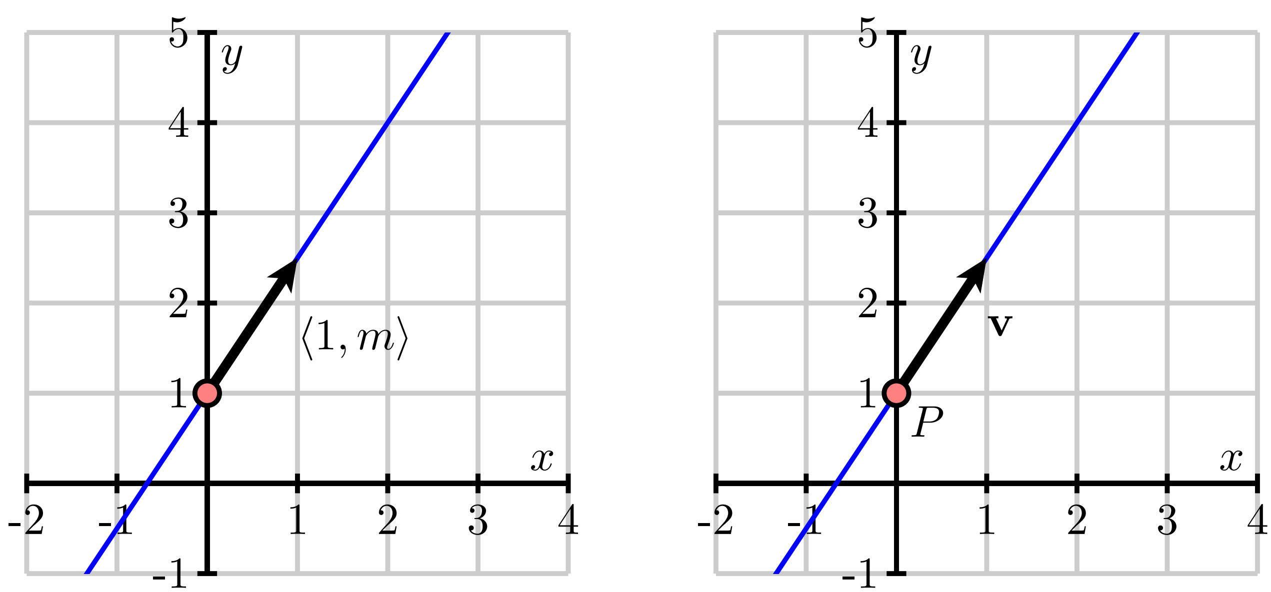


Figure 9.5.2. A vector description of a line

Definition 9.5.3. A ***line*** in space is the set of terminal points of vectors emanating from a given point P that are parallel to a fixed vector $\mathbf{v}$.

The fixed vector $\mathbf{v}$ in the definition is called a *direction vector* for the line. As we