Expansion of a Nuclear Blast

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This is an example of the final draft of a "problem writeup" from ***Thermodynamics****. But many of the writing principles are illustrated here.*

Diagrams*: Quantities in thermodynamics, like temperature and pressure, are not things you can easily picture on a diagram. But in Mechanics you all *should* be able to come up with a diagram that labels coordinates / angles in the problem you're working on.*

*Notice that he *has* included a helpful plot (with labelled axes) of radius of the blast as a function of temperature.*

Motivation: *This problem has obvious and interesting connections with a real world problem! So the "motivation" section deals with the application quite a bit. Yours may not have an obvious real-world application so won't dwell on that as much.*

But his motivation section also includes the sort of "roadmap" comments I'm looking for, like the goal of "estimating the relationship between the temperature and the radius".

Narration: *He narrates the mathematical operations he's doing. (You don't have to narrate every step of algebra, but can use some shorthand like "after solving this equation for the energy, the result is..."*

Part of narrating involves referring to the equations you're using. In this paper, he has just automatically numbered every equation.

Already your first draft should have not only equations, but also enough narration to lay out what you're trying to accomplish from step to step.
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Problem 4.15

From Carter's *Classical and Statistical Thermodynamics*:

Shortly after detonation the fireball of a uranium fission bomb consists of a sphere of radius 15 m and temperature $3 \times 10^5\ K$. Assuming that the expansion is adiabatic and that the fireball remains spherical, estimate the radius of the ball when the temperature is 3000 K. (Take $\gamma = 1.4$.)

Motivation

Nuclear weapons are designed as controlled chain reactions intended to cause different levels of damage at specific radii from the detonation site. One of the major destructive components of a nuclear blast is the fireball of heat generated by the reaction. In the pursuit of a more effective weapon which attempts to limit collateral damage it is useful to know the extent of the fireball given a certain initial heat.

The following thermodynamic analysis will result in an adequate estimation of the real radii of the blast at the given temperature, as well as demonstrating the relationship between the temperature and the radius in general. In this manner, the temperature and associated damage can be found at any radius from detonation. This information is not only useful for the development of nuclear weapons, but it is also relevant to evacuation plans and bunker engineering in the case of weaponized nuclear attacks.

Exposition

The specific heat capacity c_v, where heat is added at a constant temperature, can be defined with respect to the change in internal energy as $c_v = \left(\frac{\partial u}{\partial T}\right)_v$, which can be rearranged to

$$c_v \, dT = du \ . \qquad [1]$$

From the first law of thermodynamics, the change in internal energy can be written as $du = \delta q - \delta w$, and the change in work is defined as $\delta w = P dv$. Subsituting in for δw gives

$$du = \delta q - P dv \ . \qquad [2]$$

Substituting Equation 2 into Equation 1 and rearranging terms gives

$$\delta q = c_v \, dT + P dv \ . \qquad [3]$$

The specific heat capacity c_p, where heat is added at a constant pressure, can be defined with respect to the change in enthalpy as $c_p = \left(\frac{\partial h}{\partial T}\right)_p$, which can be rearranged to

$$c_p \, dT = dh \ . \qquad [4]$$

The enthalpy, h, is defined as $h = u + Pv$, which when differentiated yields $dh = du + P dv + v dP$. Using Equation 2 and Equation 4, δq can be written in terms of c_p as

$$\delta q = c_p \, dT - v dP \ . \qquad [5]$$

In the statement of the problem, the gas expansion is assumed to be adiabatic. This is likely an adequate assumption given the rapid speed of the expansion. For adiabatic processes, δq for the system must be equal to zero. Given this condition, Equations 3 and 5 can be rewritten as

$$c_v \, dT = -P dv \ , \qquad [6]$$

and

$$c_p \, dT = v dP \ . \qquad [7]$$

Dividing Equation 7 by Equation 6 yields

$$\frac{v dP}{P dv} = -\frac{c_p}{c_v} . \qquad [8]$$

The constant γ given in the description of the problem is defined as $\frac{c_p}{c_v}$. This can be substituted into Equation 8, and the terms can be rearranged to give

$$\frac{dP}{P} = -\gamma \frac{dv}{v} \ . \qquad [9]$$

Mathematica can be used to integrate both sides of Equation 9.

```
Integrate[1 / P, P]
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Log[P]
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Integrate[-γ / v, v]
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-γ Log[v]
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Both of these integrals are indefinite and would include a constant of integration, which can be written as one combined constant, K_0, on the right-hand side of the equation. Now Equation 9 becomes

$$\ln(P) = -\gamma \ln(v) + K_0 \ . \qquad [10]$$

Taking the exponential function of both sides yields

$$P = v^{-\gamma} \ K \ , \qquad [11]$$

where K is a new constant equal to e^{K_0}. Equation 11 rearranges to

$$P v^{\gamma} = K \ . \qquad [12]$$

Using the ideal gas law, $Pv = RT$, Equation 12 can be expressed in terms of T, v and γ, corresponding to the information given in the problem. Equation 12 becomes

$$T \ v^{\gamma-1} = K' \ , \qquad [13]$$

where K' is equal to $\frac{K}{R}$. It can be seen from Equation 13 that for the adiabatic expansion of an ideal gas, T is proportional to $v^{\gamma-1}$. Next, the equation for the volume of a sphere is πr^3. Therefore, the problem can be defined mathematically as

$$T_1 \left(r_1^{\ 3}\right)^{\gamma-1} = T_2 \left(r_2^{\ 3}\right)^{\gamma-1} \ , \qquad [14]$$

where T_1 and r_1 are the conditions shortly after detonation. The terms involving the gas constant K' and π are not seen in the equation because they are identical on both sides. The conditions given in the problem can be substituted into Equation 14 to find an equation for r_2. This substitution results in

$$\left(3 \times 10^5 \ K\right) \left(15^3 \ m\right)^{1.4-1} = (3000 \ K) \left(r_2^{\ 3}\right)^{1.4-1} \ . \qquad [15]$$

Equation 15 can now be solved in *Mathematica*.

```
Solve[300 000 * (15 ^ 3) ^ 0.4 == 3000 * (r ^ 3) ^ 0.4, r]
```

```
{{r → -348.119 - 602.96 i}, {r → -348.119 + 602.96 i}, {r → 696.238}}
```

Mathematica finds a number of solutions, but only one of them fits the real application of radius (it must be real and positive). Thus, the radius of the fireball has increased to from 15 *m* to approximately 696 *m* when the temperature has decreased from 300,000 *K* to 3000 *K*.

The Radius at Any Final Temperature

It is more useful to consider the radius of the fireball given any final temperature between the initial temperature of 3×10^5 *K* and room temperature (298 *K*). A plot of radius against temperature is one useful method of considering the this question. To achieve this plot, Equation 13 first be solved for *K'* using the given intial temperature and radius. Then the calculated value can be substituted back into

Equation 13, which now can be solved for *r* and plotted against *T*. *Mathematica* solves for *K*' below.

```
Solve[k == 300 000 * (Pi * 15 ^ 3) ^ 0.4, k]
```

$\{\{k \to 1.22262 \times 10^{7}\}\}$

Now Equation 13 becomes

$$r = \sqrt[3]{\left(\frac{1.22\times10^{7}}{T}\right)^{2.5}\left(\frac{1}{\pi}\right)}\ . \qquad [16]$$

Equation 16 can be plotted with *Mathematica* to show the relationship between the radius of the fireball and its temperature. A semi-log plot is used for better visualization.

```
LogPlot[∛((1.22 × 10^7 / T)^2.5 (1/π)), {T, 289, 300 000}, AxesLabel → {T, r}]
```

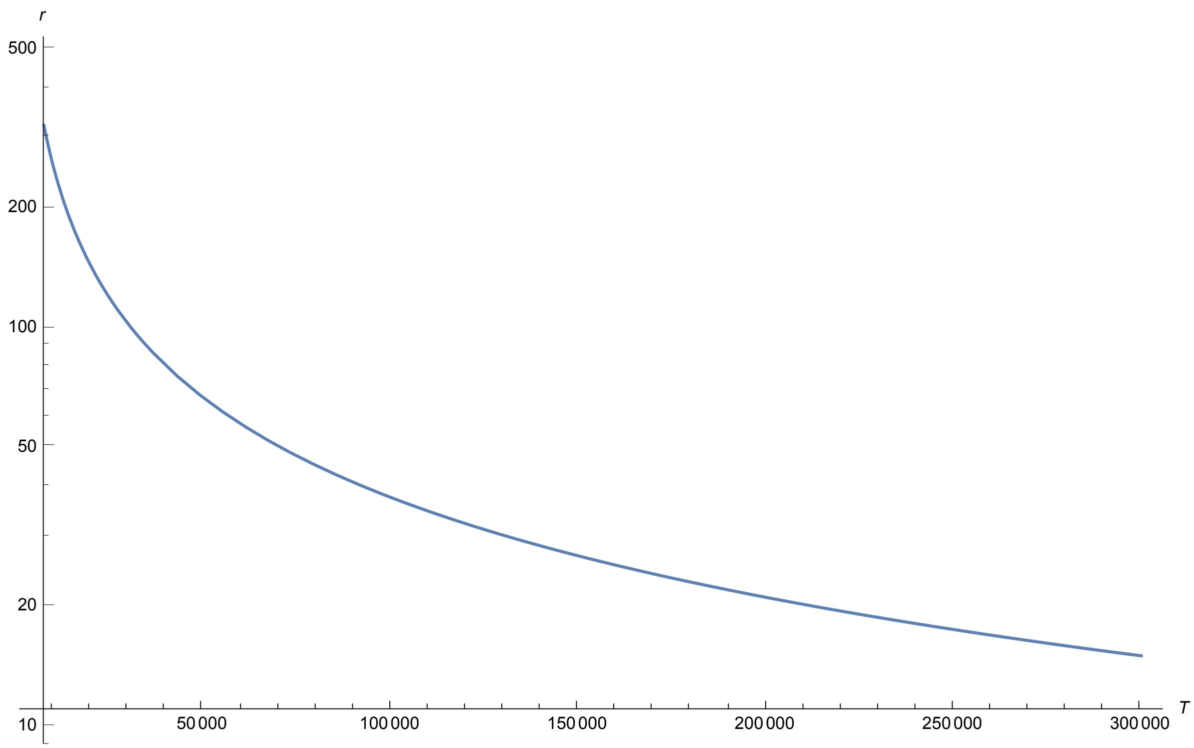
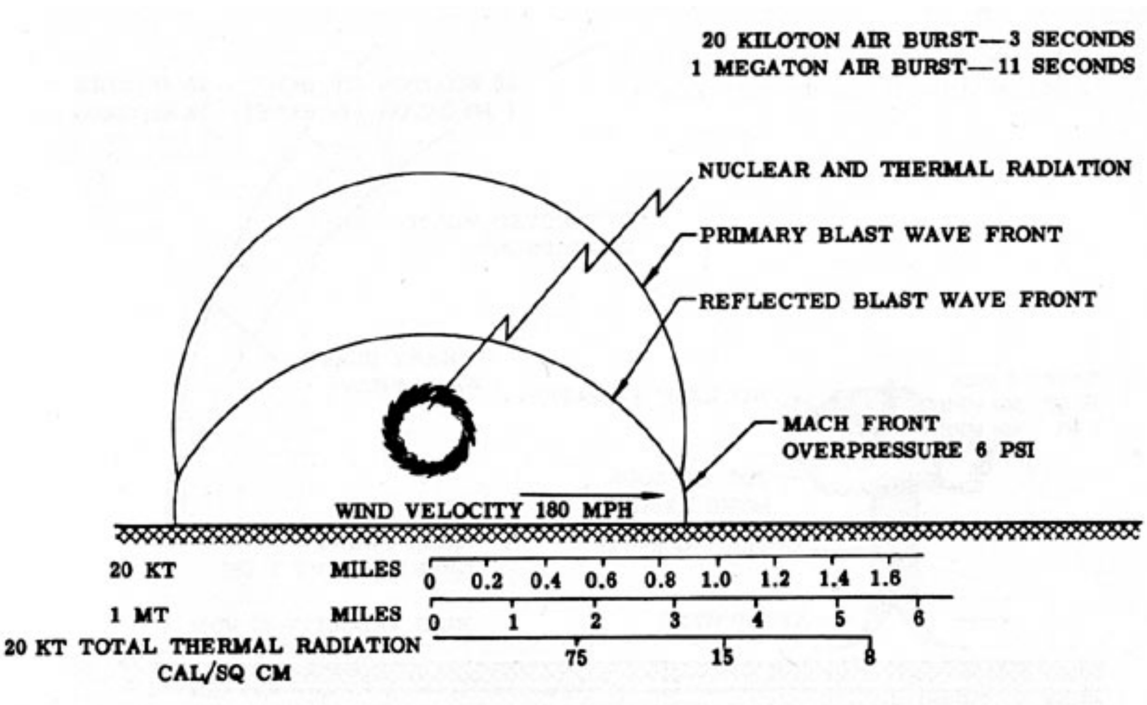


Diagram of a More Realistic Expansion for a Nuclear Blast

The problem given by Carter assumes the fireball produced by a nuclear fission bomb would be perfectly spherical, but this is highly unlikely if it is detonated on Earth. Shown below is a more realistic diagram for the expansion of the fireball, courtesy of **abomb1.org.**


20 KILOTON AIR BURST— 3 SECONDS
1 MEGATON AIR BURST— 11 SECONDS
NUCLEAR AND THERMAL RADIATION
PRIMARY BLAST WAVE FRONT
REFLECTED BLAST WAVE FRONT
MACH FRONT
OVERPRESSURE 6 PSI
WIND VELOCITY 180 MPH
20 KT
MILES
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6
1 MT
MILES
0 1 2 3 4 5 6
20 KT TOTAL THERMAL RADIATION
CAL/SQ CM
75 15 8