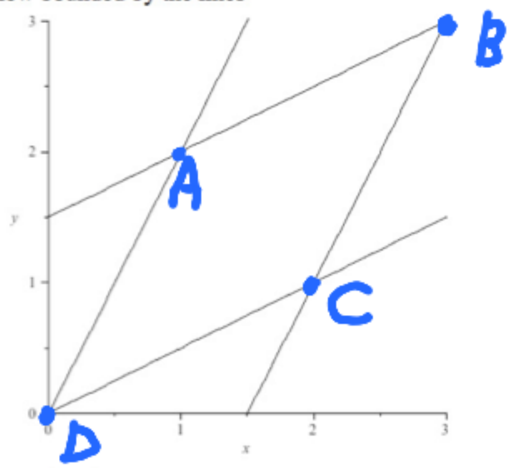
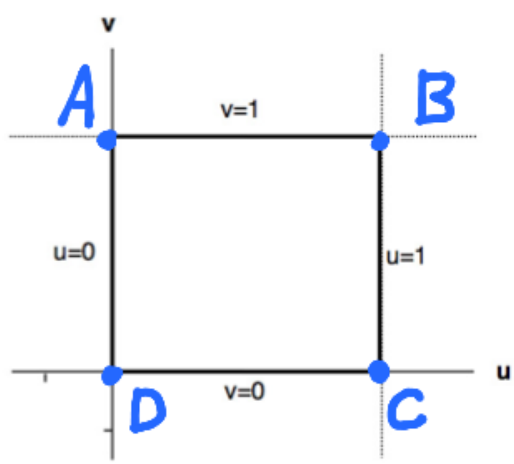
1. Consider the region pictured below bounded by the lines

$$y = \frac{1}{2}x$$
$$y = \frac{1}{2}x + \frac{3}{2}$$
$$y = 2x$$
$$y = 2x - 3$$



Find a change of variables $\begin{matrix} x = f(u,v) \\ y = g(u,v) \end{matrix}$ to transform this region into the square region shown below.



$u = 0$
$v = 0$
$u = 1$
$v = 1$

Calculate the Jacobian for that transformation.

In [18]:
```
# Find the 4 intersections of 4 lines
var('x y u v')

f1(x)=2*x
f2(x)=x/2+3/2
f3(x)=2*x -3
f4(x)=x/2

solve([f1(x)==f2(x),y==f1(x)], x,y)
```

Out[18]: `[[x == 1, y == 2]]`

In [19]:
```
# This is the point that we'd like to map to (u,v)=(0,1)
A=(1,2)

solve([f2(x)==f3(x),y==f2(x)], x,y)
```
Out[19]: `[[x == 3, y == 3]]`

In [20]:
```
# This will map to (u,v)=(1,1)
B=(3,3)

solve([f3(x)==f4(x),y==f3(x)], x,y)
```
Out[20]: `[[x == 2, y == 1]]`

In [21]:
```
# This will map to (u,v)=(1,0)
C=(2,1)

solve([f4(x)==f1(x),y==f4(x)], x,y)
```
Out[21]: `[[x == 0, y == 0]]`

In [22]:
```
# This will map to (u,v)=(0,0)
D=(0,0)
```

Now, we'd like to find a linear transformation, like...

$$x(u) = au + bv + c$$
$$y(u) = du + ev + f$$

So, using the points we have, we need to solve for the parameters a, b, c, d, e and f.

Point D implies that $(x, y) = (0, 0)$ when $(u, v) = (0, 0)$. Plugging these into the two equations above.

$$0 = a * 0 + b * 0 + c$$
$$0 = d * 0 + e * 0 + f$$

Apparently $c = 0$ and $f = 0$

Point C implies that $(x, y) = (2, 1)$ when $(u, v) = (1, 0)$:

$$2 = a * 1 + b * 0$$
$$1 = d * 1 + e * 0$$

$\Rightarrow a = 2$ and $d = 1$

Point A implies that $(x, y) = (1, 2)$ when $(u, v) = (0, 1)$

$$1 = 2 * 0 + b * 1$$
$$2 = 1 * 0 + e * 1$$

$\Rightarrow b = 1$ and $e = 2$.

We have all the constants! Let's check what happens at point B. We'd like $(x, y) = (3, 3)$ when $(u, v) = (1, 1)$:

$$x \text{ equation: } 3 = a * u + b * v + c = 2 * 1 + 1 * 1 + 0 \text{ Yes, true!}$$
$$y \text{ equation: } 3 = d * u + e * v + f = 1 * 1 + 2 * 1 + 0 \text{ Yes, true!}$$

So this transformation will accomplish the mapping:

$$x(u, v) = 2u + v,$$
$$y(u, v) = u + 2v.$$

The Jacobian is this determinant:

$$\det\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \det\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

This *should* be the ratio of areas. Clearly the area of the square $\Delta u \Delta v = 1 * 1 = 1$.

The area of the original quadrilateral is the magnitude of the cross product $\mathbf{DC} \times \mathbf{DA}$, which is hopefully 3 times as big as 1...

In [27]:
```
vDA=vector([1,2,0]) # .cross_product only works for 3 element vectors in sagemath
vDC=vector([2,1,0])
norm(   vDC.cross_product(vDA) )
```

Out[27]: 3