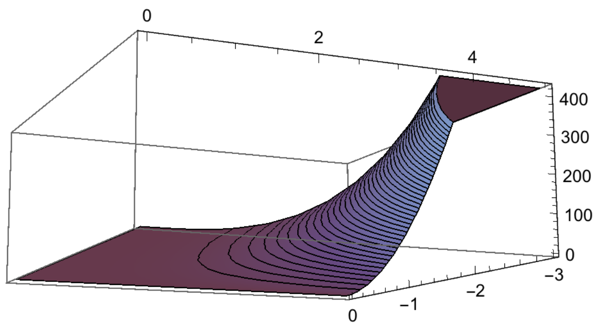
Class 08 Gradient and Optimization

Gradient

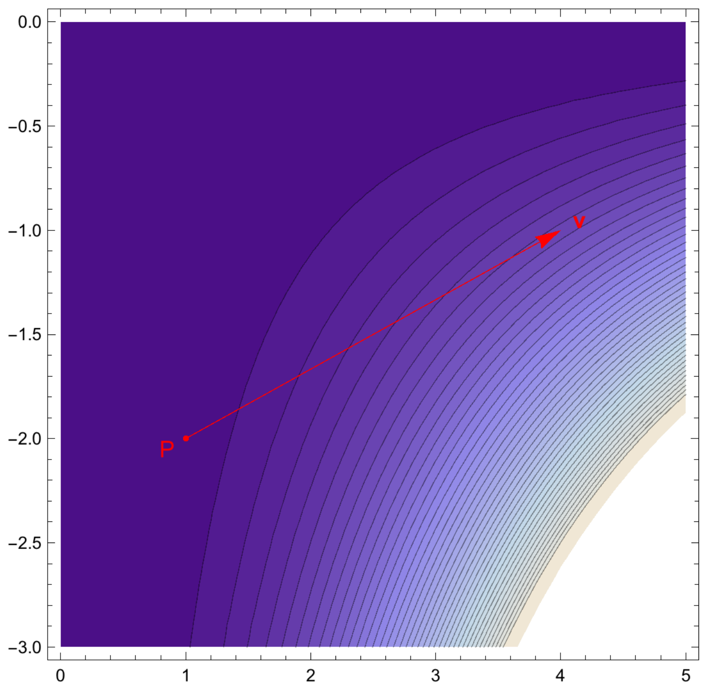
Suppose $f(x, y) = x^3 y^2$. Estimate and then find $D_{\langle 3,1\rangle} f(1, -2)$ from the definition.

$D_{\boldsymbol{v}} f(P) = \lim_{h\to 0} \frac{f(P+h\,\boldsymbol{v})-f(P)}{h}$. Call directional if $\|\boldsymbol{v}\| = 1$.

```
Plot3D[x^3 * y^2, {x, 0, 5}, {y, -3, 0}, MeshFunctions -> {#3 &}, Mesh → 40]
```



```
f[x_, y_] := x^3 y^2
a = 1; b = -2; p = {a, b};
xlo = 0; xhi = 5; ylo = -3; yhi = 0;
v = {3, 1};
plot1 = ContourPlot[f[x, y], {x, xlo, xhi}, {y, ylo, yhi}, Contours → 40];
plot2 = Graphics[{Red, Point[p], Arrow[{p, p + v}],
    Text[Style["P", 12], p - 0.05 v], Text[Style["v", 12, Bold], p + 1.05 v]}];
Show[
 plot1,
 plot2]
```




```
f[x_, y_] := x^3 y^2
(f[1 + 3 h, -2 + h] - f[1, -2])/h /. h → 1
Limit[(f[1 + 3 h, -2 + h] - f[1, -2])/h, h → 0]
```

```
60
```

```
32
```

Exercise

Find $D_{\langle 1.5, 0.5\rangle} f(1, -2)$.

```
f[x_, y_] := x^3 y^2
(f[1 + 1.5 h, -2 + 0.5 h] - f[1, -2])/h /. h → 1
Limit[(f[1 + 3 h, -2 + h] - f[1, -2])/h, h → 0]
```

31.1563

32

Let $P=(a, b)$ and $\boldsymbol{v}=\langle v_x, v_y\rangle$. Then
$\frac{f(P+h\,\boldsymbol{v})-f(P)}{h}=\frac{f(a+h\,v_x,\,b+h\,v_y)-f(a,b)}{h}$

Hence, $f_x(P)=D_{\langle 1,0\rangle} f(P)$

Use the linear approximation to write

$$\frac{f(a+h\,v_x,\,b+h\,v_y)-f(a,b)}{h}$$
$$\approx \frac{f(a,b)+f_x(a,b)\,(a+h\,v_x-a)+f_y(a,b)\,(b+h\,v_y-b)-f(a,b)}{h}$$
$$=\frac{f_x(a,b)\,(h\,v_x)+f_y(a,b)\,(h\,v_y)}{h}$$
$$=f_x(a,b)\,v_x+f_y(a,b)\,v_y$$
$$=\nabla f_{(a,b)}\cdot \boldsymbol{v}$$

If the error in the linear approximation goes to zero faster than h (this is what Rogawski calls “local linearity” and is true if the first partial derivatives are continuous), then $D_{\boldsymbol{v}} f(P)=\nabla f(P)\cdot \boldsymbol{v}$.

Exercise

Find $D_{\langle 3,1\rangle} f(1,-2)$ from the above formula.

```
Grad[f[x, y], {x, y}].{3, 1} / Sqrt[10] /. {x → 1, y → -2}
```

$16\sqrt{\frac{2}{5}}$

```
f[x_, y_] := x^3 y^2
{Derivative[1, 0][f][1, -2], Derivative[0, 1][f][1, -2]}.{3, 1}
```

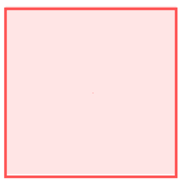
32

$\|D_{\boldsymbol{v}} f(P)\| = \|\nabla f(P)\|\ \|\boldsymbol{v}\|\ |\cos\theta|$, where θ is the angle between the direction $\boldsymbol{v}$ and the gradient $\nabla f(P)$. Thus, the directional derivative is maximized when $\boldsymbol{v}=\nabla f(P)$.

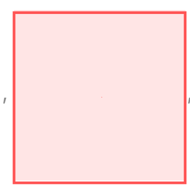
```
f[x_, y_] := x^3 y^2;
a = 1; b = -2; p = {a, b};
xlo = 0; xhi = 2; ylo = -3; yhi = -1;
plot1 = ContourPlot[f[x, y], {x, xlo, xhi}, {y, ylo, yhi}, Contours → 40];
Manipulate[
 v = {Cos[θ], Sin[θ]};
 plot2 = Graphics[{Red, Point[p], Arrow[{p, p + v}]}];
 dvf = N[{Derivative[1, 0][f][a, b], Derivative[0, 1][f][a, b]}.v];
 Show[plot1, plot2, PlotLabel -> If[show, "Dᵥf(P)  = " <> ToString[dvf], ""]],
 {θ, 0, 2 π},
 {{show, False}, {True, False}}]
```

θ

show

Show[plot1, ,

PlotLabel → $D_v f(P)$ = $-0.33282\, f^{(0,1)}[a, b] + 0.942991\, f^{(1,0)}[a, b]$]

Show: Could not combine the graphics objects in Show[plot1, , PlotLabel → $D_v f(P)$ = $-0.33282\, f^{(0,1)}[a, b] + 0.942991\, f^{(1,0)}[a, b]$].

```
Grad[f[x, y], {x, y}] /. {x → 1, y → -2}
```

{12, -4}

```
f[x_, y_] := x^3 y^2;
GD := Grad[f[x, y], {x, y}] /. {x → 1, y → -2}
```

```
GradDirection
```

{12, -4}

```
GD
```

{12, -4}

```
N[ArcTan[12, -4] + 2 * Pi] / Pi * 180
```

341.565

```
N[A]
```

-0.321751

```
N[A + 2 * Pi]
```

5.96143

```
N[(A + 2 * Pi) / Pi * 180.]
```

341.565

Gradient and Motion

Suppose the density of an object x cm from the back, y cm from the left side, and z cm from the bottom is given by $f(x, y, z) = x\,y + 3\,z$ g/cm^2.

Find $f_y(2, 1, 3)$ and interpret in words.

Find $D_{\langle 1/3, 2/3, 2/3\rangle} f(2, 1, 3)$ and interpret in words.

Find $\frac{\partial f}{\partial \rho}$ where ρ is a spherical coordinate, and interpret is words.

If you started at (2, 1, 3) and moved in the direction of greatest density at a speed proportional to the density, what would be the path of your motion?

```
f[x_, y_, z_] := x y + 3 z
Derivative[0, 1, 0][f][2, 1, 3]
```

2

If you started at (2, 1, 3) and moved in the direction of greatest density at a speed proportional to the density, what would be the path of your motion? To answer this question, we need the gradient at arbitrary points.

```
grad = {D[f[x, y, z], x], D[f[x, y, z], y], D[f[x, y, z], z]}
```

{y, x, 3}

The time derivatives of the spacial variables must be proportional to the gradient. This results in a differential equation.

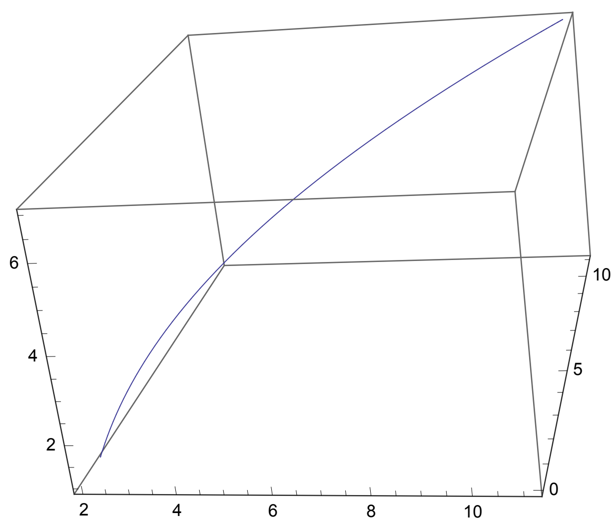
```
DSolve[{x'[t] == y[t], y'[t] == x[t], z'[t] == 3, x[0] == 2, y[0] == 1, z[0] == 1},
 {x[t], y[t], z[t]}, t]
```

$\left\{\left\{x[t] \to \frac{1}{2} e^{-t} \left(1 + 3 e^{2 t}\right), y[t] \to \frac{1}{2} e^{-t} \left(-1 + 3 e^{2 t}\right), z[t] \to 1 + 3 t\right\}\right\}$

```
PathVector
```

$$\left\{\left\{x[t]\to \frac{1}{2}e^{-t}\left(1+3e^{2t}\right),\ y[t]\to \frac{1}{2}e^{-t}\left(-1+3e^{2t}\right),\ z[t]\to 1+3t\right\}\right\}$$

```
ParametricPlot3D[{1/2 e^-t (1 + 3 e^(2 t)), 1/2 e^-t (-1 + 3 e^(2 t)), 1 + 3 t}, {t, 0, 2}]
```

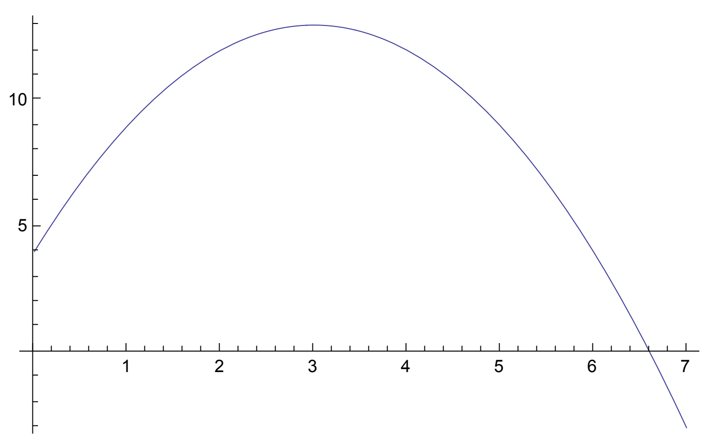


Optimization

Compare and contrast finding maxima and minima of single variable and multiple variable real-valued functions.

Find the maximum and minimum of $f(x) = 4 + 6x - x^2$ on the interval [0, 7]. Do by hand first. Include the 2nd derivative test.

```
f[x_] := 4 + 6 x - x^2
Plot[f[x], {x, 0, 7}]
```



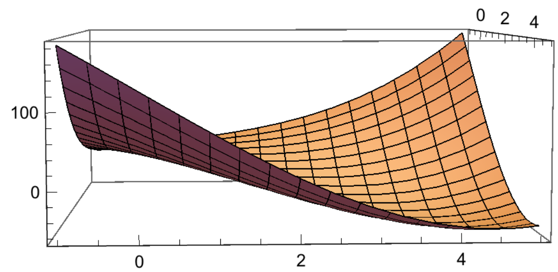
```
Minimize[{f[x], 0 ≤ x ≤ 7}, x]
Maximize[{f[x], 0 ≤ x ≤ 7}, x]
```

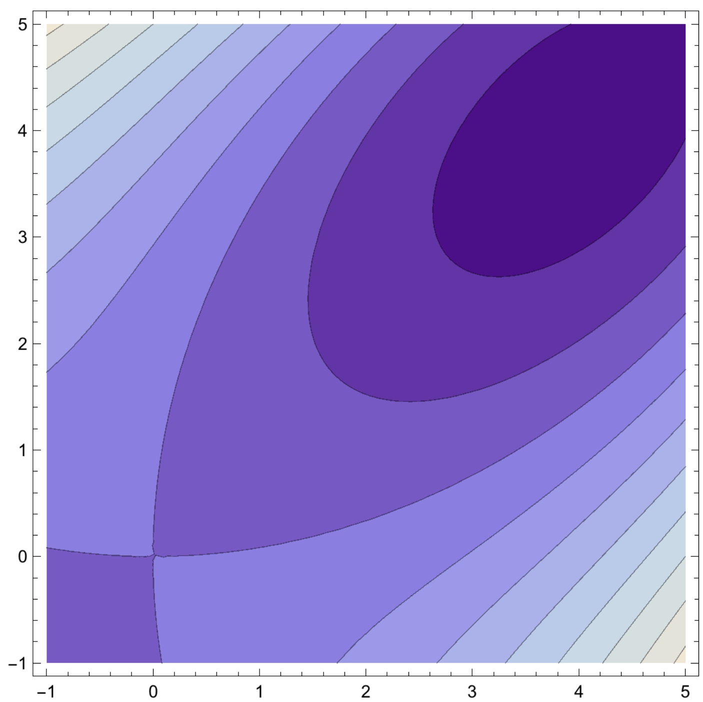
```
{-3, {x → 7}}
```

```
{13, {x → 3}}
```

Find the maximum and minimum of $f(x, y) = x^3 + y^3 - 12\,x\,y$ on the square $-1 \le x \le 5$, $-1 \le y \le 5$. Do by hand first. Include the 2nd derivative test.

```
f[x_, y_] := x^3 + y^3 - 12 x y
Plot3D[f[x, y], {x, -1, 5}, {y, -1, 5}]
ContourPlot[f[x, y], {x, -1, 5}, {y, -1, 5}]
```






```
f[x_, y_] := x^3 + y^3 - 12 x y
Minimize[{f[x, y], -1 ≤ x ≤ 5, -1 ≤ y ≤ 5}, {x, y}]
Maximize[{f[x, y], -1 ≤ x ≤ 5, -1 ≤ y ≤ 5}, {x, y}]
```

```
{-64, {x → 4, y → 4}}
```

```
{184, {x → -1, y → 5}}
```

Give an example of a function f with the property that $f_x(a, b) = f_y(a, b) = 0$ but f does not have a local maximum or minimum at (a, b).

The previous example $f(x, y) = x^3 + y^3 - 12\,x\,y$ at $(0, 0)$.